A note on the series representation for the density of the supremum of a stable process

Hackmann Daniel[*]   Kuznetsov Alexey[†‡]

Abstract

An absolutely convergent double series representation for the density of the supremum of α-stable Lévy process is given in [3, Theorem 2] for almost all irrational α. This result cannot be made stronger in the following sense: the series does not converge absolutely when α belongs to a certain subset of irrational numbers of Lebesgue measure zero (see [6, Theorem 2]). Our main result in this note shows that for *every* irrational α there is a way to rearrange the terms of the double series, so that it converges to the density of the supremum. We show how one can establish this stronger result by introducing a simple yet non-trivial modification in the original proof of [3, Theorem 2].

1 Introduction

Consider an α-stable Lévy process X, defined by the characteristic exponent

$$\Psi(z) = -\ln \mathbb{E}\left[e^{izX_1}\right] = |z|^\alpha \times \left(e^{\pi i \alpha(\frac{1}{2}-\rho)}\mathbf{1}_{\{z>0\}} + e^{-\pi i \alpha(\frac{1}{2}-\rho)}\mathbf{1}_{\{z<0\}}\right), \quad z \in \mathbb{R}. \quad (1.1)$$

Here (α, ρ) belong to the set of admissible parameters

$$\mathcal{A} = \{\alpha \in (0,1), \ \rho \in (0,1)\} \cup \{\alpha \in (1,2), \ \rho \in [1-\alpha^{-1}, \alpha^{-1}]\}.$$

It can be shown (see [6] and Section 4.6 in [8]) that with this parameterization we have $\rho = \mathbb{P}(X_1 > 0)$ and that any stable process with $\alpha \neq 1$ can be rescaled so that it's characteristic exponent is written in the form (1.1).

Our main object of interest is the density $p(x)$ of the supremum

$$S_1 := \sup\{X_u \ : 0 \le u \le 1\}.$$

Let us summarize what is currently known about this function.

---

[*]Department of Mathematics and Statistics, York University, Toronto, Canada;
  E-mail: dhackman@mathstat.yorku.ca, URL: www.danhackmann.com
[†]Department of Mathematics and Statistics, York University, Toronto, Canada;
  E-mail: kuznetsov@mathstat.yorku.ca, URL: www.math.yorku.ca/~akuznets
[‡]Research supported by the Natural Sciences and Engineering Research Council of Canada.

(i) The simplest case is when X belongs to one of the Doney classes $\mathcal{C}_{k,l}$ (here $k, l \in \mathbb{Z}$), which are defined by the relationship $\rho + k = l/\alpha$ (see [2]). In this case an absolutely convergent series representation for $p(x)$ is given in [5, Theorem 10]. From now on we assume that X does not belong to one of the Doney classes.

(ii) When α is rational, [6, Theorem 3] gives an explicit formula for the Mellin transform of $p(x)$, which is expressed in terms of elementary functions and the dilogarithm function $\mathrm{Li}_2(z)$. Unfortunately, there seems to be little hope of obtaining an explicit series representation for $p(x)$: This problem is equivalent to evaluating the residues of the Mellin transform, which in this case has multiple poles and computing these residues in closed form seems to be impossible.

(iii) When α is irrational, we define sequences $\{a_{m,n}\}_{m \geq 0, n \geq 0}$ and $\{b_{m,n}\}_{m \geq 0, n \geq 1}$ as follows

$$
\begin{aligned}
a_{m,n} := {} & \frac{(-1)^{m+n}}{\Gamma\left(1 - \rho - n - \frac{m}{\alpha}\right)\Gamma(\alpha\rho + m + \alpha n)} \\
& \times \prod_{j=1}^{m} \frac{\sin\left(\frac{\pi}{\alpha}(\alpha\rho + j - 1)\right)}{\sin\left(\frac{\pi j}{\alpha}\right)} \prod_{j=1}^{n} \frac{\sin(\pi\alpha(\rho + j - 1))}{\sin(\pi\alpha j)},
\end{aligned}
\tag{1.2}
$$

$$
b_{m,n} := \frac{\Gamma\left(1 - \rho - n - \frac{m}{\alpha}\right)\Gamma(\alpha\rho + m + \alpha n)}{\Gamma\left(1 + n + \frac{m}{\alpha}\right)\Gamma(-m - \alpha n)} a_{m,n}.
\tag{1.3}
$$

As was established in [3], there exists a set of irrational numbers $\mathcal{L}$, which is uncountable, dense, and has Lebesgue measure zero, such that for all irrational $\alpha \notin \mathcal{L}$ we have

$$
p(x) = x^{-1-\alpha} \sum_{n \geq 0} \sum_{m \geq 0} b_{m,n+1} x^{-m-\alpha n}, \quad \text{if } \alpha \in (0,1),
\tag{1.4}
$$

$$
p(x) = x^{\alpha\rho-1} \sum_{n \geq 0} \sum_{m \geq 0} a_{m,n} x^{m+\alpha n}, \qquad \text{if } \alpha \in (1,2).
\tag{1.5}
$$

(iv) Theorem 2 in [6] states that the previous result cannot be substantially improved: There exists an uncountable dense subset $\tilde{\mathcal{L}} \subset \mathcal{L}$, such that for all $\alpha \in \tilde{\mathcal{L}}$ and almost all ρ the series in (1.4), (1.5) do not converge absolutely for all $x > 0$.

This situation is clearly deficient, since we do not have a useful expression for $p(x)$ if $\alpha \in \mathcal{L}$ and we do not know whether the series (1.4), (1.5) converge if $\alpha \in \mathcal{L} \setminus \tilde{\mathcal{L}}$. Also, determining whether α belongs to $\mathcal{L}$ or $\tilde{\mathcal{L}}$ is problematic, as this would require full knowledge of the continued fraction representation of α (see [3, Proposition 1] and [6, Proposition 1]). Since it is impossible to have absolute convergence of the series for all irrational α, our only remaining possibility is to try to find some sort of "conditional" convergence. Absolute convergence implies that the order of summation does not matter. In the absence of absolute convergence, the way in which we compute the partial sums of the double series becomes very important. Our main result shows that the right way to compute the partial sums in (1.4), (1.5) is over triangles $\{m + \alpha n < C : m \geq 0, n \geq 0\} \subset \mathbb{Z}^2$, and that one can always find an increasing sequence $C_k \to +\infty$ (which depends crucially on the arithmetic properties of α), such that the partial sums will converge to $p(x)$.

2 Main result

For $x \in \mathbb{R}$, let $[x]$ denote the largest integer not greater than x and let $\{x\} := x - [x]$ denote the fractional part of x. The continued fraction representation (see [4]) is defined

as follows

$$x = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}$$

where $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for $i \geq 1$. The coefficients of the continued fraction can be computed recursively in the following way: Define $x_1 := \{x\}$ and $x_{i+1} := \{1/x_i\}$, $i \geq 1$, then $a_0 = [x]$ and $a_i = [1/x_i]$, $i \geq 1$. For $x \notin \mathbb{Q}$ the continued fraction has infinitely many terms; truncating it after n steps results in a rational number $p_n/q_n := [a_0; a_1, a_2, ..., a_n]$, which provides the best rational approximation to x (among all rational numbers with denominators not greater than q_n) and is called the n-th convergent (see [4, Theorem 17]). The numerators and denominators of convergents are known to satisfy the two-term recurrence relation

$$\begin{cases} p_n = a_n p_{n-1} + p_{n-2}, & p_{-1} = 1, \quad p_{-2} = 0, \\ q_n = a_n q_{n-1} + q_{n-2}, & q_{-1} = 0, \quad q_{-2} = 1. \end{cases} \tag{2.1}$$

The following theorem is our main result in this note.

Theorem 2.1. *Assume that $\alpha \notin \mathbb{Q}$. Then for all $x > 0$*

$$p(x) = \begin{cases} x^{-1-\alpha} \displaystyle\lim_{k \to \infty} \sum_{\substack{m+1+\alpha(n+\frac{1}{2})<q_k \\ m \geq 0,\, n \geq 0}} b_{m,n+1} x^{-m-\alpha n}, & \text{if } \alpha \in (0,1), \\[4em] x^{\alpha\rho-1} \displaystyle\lim_{k \to \infty} \sum_{\substack{m+1+\alpha(n+\frac{1}{2})<q_k \\ m \geq 0,\, n \geq 0}} a_{m,n} x^{m+\alpha n}, & \text{if } \alpha \in (1,2), \end{cases} \tag{2.2}$$

where $a_{m,n}$ and $b_{m,n}$ are defined by (1.2) and (1.3) and $q_k = q_k(2/\alpha)$ is the denominator of the k-th convergent for $2/\alpha$.

To obtain the proof of Theorem 2.1 one would need to modify only one step in the proof of [3, Theorem 2]. Instead of reproducing the long and technical proof of the latter result in full entirety, we only explain here which step has to be modified.

Assume that $\alpha \in (1,2)$ and that $\alpha \notin \mathbb{Q}$. Combining formulas (17), (21), (24) and (26) in [3] we find that for every integer $k \geq 1$

$$p(x) = x^{\alpha\rho-1} \sum_{\substack{m+\alpha(n+\frac{1}{2})<k \\ m \geq 0,\, n \geq 0}} a_{m,n} x^{m+\alpha n} + \mathsf{e}_k(x), \tag{2.3}$$

where the error term $\mathsf{e}_k(x)$ satisfies

$$|\mathsf{e}_k(x)| < (A(1+x))^k \times e^{-\epsilon k \ln(k)} \times \prod_{l=1}^{k} \left| \sec\left(\frac{\pi l}{\alpha}\right) \right|, \tag{2.4}$$

for some constants $A > 0$ and $\epsilon > 0$ (which can depend on (α, ρ) but not on x or k). The main step in the proof [3, Theorem 2] is to construct a set $\mathcal{L} \subset \mathbb{R} \setminus \mathbb{Q}$ (which is dense, uncountable, of Lebesgue measure zero), such that for all $\alpha \notin \mathcal{L}$ we have an upper bound

$$\prod_{l=1}^{k} \left| \sec\left(\frac{\pi l}{\alpha}\right) \right| < B3^k, \quad k \geq 1, \tag{2.5}$$

for some constant $B = B(\alpha)$. Combining (2.4) with (2.5) implies that $\mathrm{e}_k(x) \to 0$ as $k \to +\infty$, which gives us

$$p(x) = x^{\alpha\rho-1} \lim_{k \to +\infty} \sum_{\substack{m+\alpha(n+\frac{1}{2}) < k \\ m \geq 0,\, n \geq 0}} a_{m,n} x^{m+\alpha n}. \tag{2.6}$$

To obtain the series representation (1.5) we use formula (1.2) and Proposition 1 and Lemma 1 in [3] and check that for $\alpha \notin \mathcal{L}$ the series (1.5) converges absolutely, therefore the order of summation in (2.6) does not matter, and the sum can be rewritten in the form (1.5).

The assumption that $\alpha \notin \mathcal{L}$ is crucial for deriving inequality (2.5) and formula (2.6). The upper bound (2.5) is not true for all irrational α, and for a suitable α the product in (2.5) can grow arbitrarily fast. The following example illustrates this phenomenon.

Example 1: We define τ via its continued fraction representation

$$\tau = [a_0; a_1, a_2, \dots] = [1, 2, 2^4, 2^{1089}, \dots]$$

where the coefficients a_n are defined as $a_{n+1} = 2^{q_n^2}$, $n \geq -1$ and the numerators p_n and the denominators q_n of the n-th convergent are computed recursively via (2.1). We find the first few terms of p_n and q_n to be

$$[p_0, p_1, p_2, \dots] = [1, 3, 49, \dots], \qquad [q_0, q_1, q_2, \dots] = [1, 2, 33, \dots].$$

Let us take $\alpha = 2/\tau \approx 1.34693878....$ Since a_n are even integers for $n \geq 1$, it follows from (2.1) that p_n are all odd numbers, so that we can write $p_n = 2r_n + 1$ for some integers r_n. From Theorem 13 in [4] we know that

$$|q_n \tau - p_n| < \frac{1}{q_{n+1}} = \frac{1}{a_{n+1}q_n + q_{n-1}} < \frac{1}{a_{n+1}} = 2^{-q_n^2},$$

therefore $|q_n/\alpha - r_n - 1/2| < 2^{-q_n^2}$. Using the inequality $|\cos(\pi x)| \leq \pi|x - 1/2|$, we conclude that

$$\left| \sec\left(\frac{\pi q_n}{\alpha} \right) \right| = \frac{1}{|\cos\left(\pi \left(\frac{q_n}{\alpha} - r_n \right) \right)|} \geq \frac{1}{\pi|\frac{q_n}{\alpha} - r_n - \frac{1}{2}|} > \frac{2^{q_n^2}}{\pi},$$

therefore for this particular choice of α the full sequence of products in (2.5) cannot be bounded from above by any exponential function of k.

As the previous example demonstrates, one can construct α, so that the full sequence of products in (2.5) can grow arbitrarily fast. What is remarkable is that for any irrational α there is always an infinite subsequence of these products which are bounded from above by an exponential function of the index.

Lemma 2.2. *Assume that $\tau \notin \mathbb{Q}$ and $\tau > 0$. There exists a constant $C = C(\tau) > 0$ such that for all $k \geq 1$*

$$\prod_{l=1}^{q_k-1} |\sec(\pi l \tau)| \quad \leq C 6^{q_k}. \tag{2.7}$$

where $q_k = q_k(2\tau)$ is the denominator of the k-th convergent for 2τ.

Proof. We use the following result (see [1, Lemma 4] or [7, Lemma 4]): for any $\beta > 0$, $\beta \notin \mathbb{Q}$,

$$\lim_{k \to \infty} \frac{1}{q_k} \sum_{l=1}^{q_k-1} \ln(2|\sin(\pi l \beta)|) = 0, \tag{2.8}$$

where $q_k = q_k(\beta)$. This is equivalent to

$$\lim_{k \to \infty} \left[\prod_{l=1}^{q_k-1} \frac{1}{|\sin(\pi l \beta)|} \right]^{\frac{1}{q_k}} = 2,$$

which implies the existence of a constant $C = C(\beta) > 0$, such that for all $k \geq 1$

$$\prod_{l=1}^{q_k-1} \frac{1}{|\sin(\pi l x)|} < C3^{q_k}.$$

Using the identity $\sin(\pi l \beta) = 2\sin(\pi l \frac{\beta}{2})\cos(\pi l \frac{\beta}{2})$ we conclude

$$\prod_{l=1}^{q_k-1} \frac{1}{|\cos(\pi l \frac{\beta}{2})|} < C3^{q_k} 2^{q_k-1} \prod_{l=1}^{q_k-1} |\sin(\pi l \frac{\beta}{2})| < C6^{q_k}.$$

Taking $\beta = 2\tau$ we obtain (2.7). $\qquad\square$

Combining (2.7) with (2.3) and (2.4) gives us the statement of Theorem 2.1 for $\alpha \in (1, 2)$. The proof for $\alpha \in (0, 1)$ can be obtained in a similar way.

References

[1] V. I. Buslaev, *On the convergence of the Rogers-Ramanujan continued fraction*, Mat. Sb. **194** (2003), no. 6, 43–66. MR-1992176

[2] R. A. Doney, *On Wiener-Hopf factorisation and the distribution of extrema for certain stable processes*, Ann. Probab. **15** (1987), no. 4, 1352–1362. MR-905336

[3] F. Hubalek and A. Kuznetsov, *A convergent series representation for the density of the supremum of a stable process*, Electron. Commun. Probab. **16** (2011), 84–95. MR-2763530

[4] A. Ya. Khinchin, *Continued fractions*, The University of Chicago Press, Chicago, Ill.-London, 1964. MR-0161833

[5] A. Kuznetsov, *On extrema of stable processes*, Ann. Probab. **39** (2011), no. 3, 1027–1060. MR-2789582

[6] _____, *On the density of the supremum of a stable process*, Stochastic Process. Appl. **123** (2013), no. 3, 986–1003. MR-3005012

[7] G. Petruska, *On the radius of convergence of q-series*, Indag. Math. (N.S.) **3** (1992), no. 3, 353–364. MR-1186743

[8] V. M. Zolotarev, *One-dimensional stable distributions*, Translations of Mathematical Monographs, vol. 65, American Mathematical Society, Providence, RI, 1986. MR-854867