The density of the supremum of a stable process

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Overview

- Introduce Lévy processes and stable processes

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- Introduce the problem, the history and a general approach

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- Introduce the problem, the history and a general approach
- Present a solution which has an interesting connection to the continued fraction representation of irrational numbers

Outline

Definition

A Lévy process is an $\mathbb{R}$-valued stochastic process $X = \{X_t : t \geq 0\}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that possesses the following properties:

(i) The paths of X are right continuous with left limits $\mathbb{P}$-a.s.

(ii) $X_0 = 0$ $\mathbb{P}$-a.s.

(iii) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

(iv) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s}.

Applications

Lévy processes are popular because they are general enough to represent real world phenomena, but tractable enough so that we may obtain meaningful results.

We find applications of Lévy processes in many fields, the natural sciences, operations management, actuarial science, and mathematical finance.

One-sided processes

Definition

Processes which are almost surely increasing are called *subordinators*. Processes which are not subordinators but have no negative jumps are called *spectrally positive*.

Infinite divisibility

A random variable ξ is *infinitely divisible* if for any $n \in \mathbb{N}$ it satisfies

$$\xi \overset{d}{=} \xi_1^n + \ldots + \xi_n^n,$$

for i.i.d. $\{\xi_i^n : i = 1, \ldots n\}$. For a Lévy process X we have

$$X_t = \left(X_t - X_{\frac{(n-1)t}{n}}\right) + \ldots + \left(X_{\frac{2t}{n}} - X_{\frac{t}{n}}\right) + X_{\frac{t}{n}},$$

and so, by the independent and stationary increments property, we have that X_t, in particular X_1, is infinitely divisible. Conversely, for any inf. div. random variable ξ, we can find a Lévy process X such that $X_1 \overset{d}{=} \xi$.

Outline

Stable processes

A *(strictly) stable random variable* is defined as a random variable which, for some $\alpha \in (0, 2]$ satisfies

$$n^{\alpha}\xi \stackrel{d}{=} \xi_1 + \ldots + \xi_n, \quad n \in \mathbb{N},$$

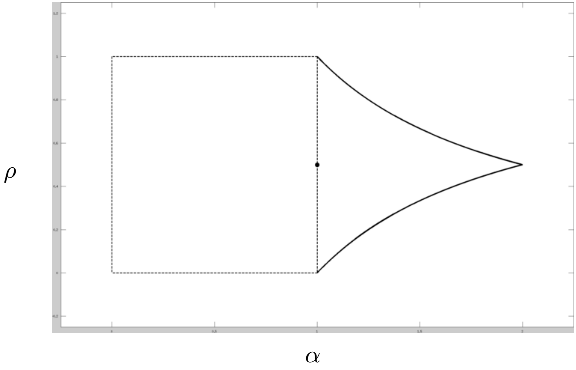
where the ξ_i are independent and distributed like ξ. Therefore, stable random variables are infinitely divisible, and so, we can speak also about stable Lévy processes X.

A natural way to categorize stable processes is by the parameters α and $\rho = \mathbb{P}(X_1 > 0)$.

Admissible parameters

We will study processes with parameters in the admissible set

$$\mathcal{A} = \{\alpha \in (0,1), \rho \in (0,1)\} \cup \{\alpha = 1, \rho = \tfrac{1}{2}\} \cup \{\alpha \in (1,2), \rho \in [1-\alpha^{-1}, \alpha^{-1}]\}.$$



Stable processes

A key feature of such processes is the *self-similarity* property:

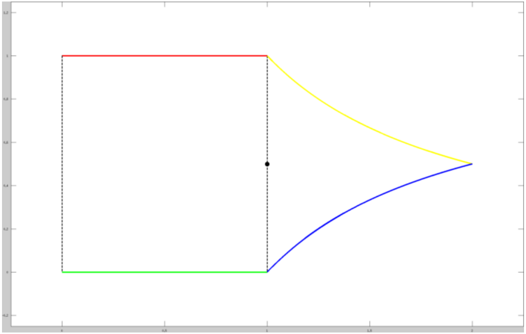
$$\{\lambda^{1/\alpha} X_t : t \geq 0\} \stackrel{d}{=} \{X_{\lambda t} : t \geq 0\}, \quad \lambda > 0,$$

i.e. scaling the process in space is equivalent to a scaling in time. A standard example: the well known scaling of Brownian motion i.e. $\sqrt{t} B_1 \stackrel{d}{=} B_t$.

More on the parameters

subordinator $\Leftrightarrow \rho = 1$,	spectrally negative $\Leftrightarrow \rho = 1/\alpha$
-subordinator $\Leftrightarrow \rho = 0$	spectrally positive $\Leftrightarrow \rho = 1 - 1/\alpha$

ρ



α

The Doney classes

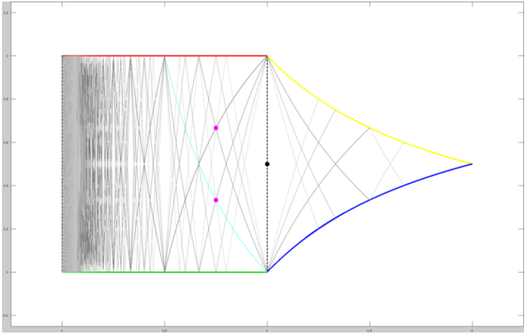
If we generalize the the spectrally negative, $\rho = 1/\alpha$, and spectrally positive, $\rho = 1 - 1/\alpha$, cases we get the processes in the Doney classes $\{\mathcal{C}_{k,l}\}_{k,l\in\mathbb{Z}}$. A stable process $X \in \mathcal{C}_{k,l}$ for some $k, l \in \mathbb{Z}$ if its parameters statisfy

$$\rho = \frac{l}{\alpha} - k.$$

The Doney classes

- The line corresponds to $X \in \mathcal{C}_{1,1}$.
- $\alpha = 3/4$ corresponds to exactly 2 possible ρ such that the process with parameters (α, ρ) is in a Doney class (see •). In general, $\alpha = m/n$ can correspond to exactly $m-1$ parameters ρ such that X is in a Doney class. These have the form $\mathcal{C}(\alpha) = \{j/m : j = 1, 2, \ldots, m-1\}$.

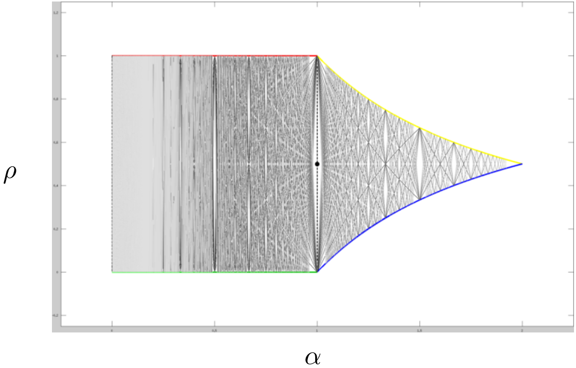
ρ



α

The Doney classes

The Doney classes correspond to a dense set of parameters.



Outline

The problem

For a stable process with admissible parameters define

$$S_t := \sup_{0 \le s \le t} X_t,$$

which is known as the *running supremum process*.

Q: Can we find an explicit expression for the density $p(x)$ of S_1?

The general problem

A: In some special cases:

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- Kuznetsov 2011: a full asymptotic expansion at 0 and ∞ and an absolutely convergent series representation for $X \in \mathcal{C}_{k,l}$
- Hubalek and Kuznetsov 2011: an absolutely convergent series for *almost* all processes where $\alpha \notin \mathbb{Q}$

The approach to the general problem

The key is to build a connection with the positive Wiener-Hopf factor of the underlying process. For a stable process X with supremum process S this is defined as

$$\varphi(z) := \mathbb{E}[\exp(-zS_{\mathbf{e}(1)})], \quad \mathrm{Re}(z) \geq 0,$$

where $S_{\mathbf{e}(1)}$ is the running supremum evaluated at $\mathbf{e}(1)$ an independent exponential random variable with mean 1.

For a general process it is not easy to obtain an explicit expression for $\varphi(z)$, but for stable processes we now have this information due to Darling 1956 ($\rho = 1/2$), Doney 1987 ($X \in \mathcal{C}_{k,l}$), and Kuznetsov 2011 (general formula).

The approach

The Mellin transforms

$$\mathcal{M}(S_1, w) := \mathbb{E}[S_1^{w-1}] = \int_{\mathbb{R}^+} x^{w-1} p(x) \mathrm{d}x, \quad 1 - \alpha\rho < \mathrm{Re}(w) < 1 + \alpha,$$

and

$$\Phi(w) := \int_{\mathbb{R}^+} z^{w-1} \varphi(z) \mathrm{d}z, \quad 0 < \mathrm{Re}(w) < \alpha\rho$$

of the two functions are related by the identity.

$$\Phi(w) = \Gamma(w)\Gamma\left(1 - \frac{w}{\alpha}\right) \mathcal{M}(S_1, 1 - w), \quad 0 < \mathrm{Re}(w) < \alpha\rho.$$

The approach

Since Kuznetsov 2011 we know $\mathcal{M}(S_1, w)$ explicitly. The question is can we invert to obtain $p(x)$?

- For $X \in \mathcal{C}_{k,l}$ the answer is yes for all pairs $(\alpha, \rho) \to$ abs. conv. series.

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- Analytical inversion when $\alpha \in \mathbb{Q}$ and $X \notin \mathcal{C}_{k,l}$ seems difficult, since $\mathcal{M}(S_1, w)$ has poles of order greater than one and computing residues seems impossible. However, Kuznetsov 2013 demonstrates that numerical inversion is accurate and reasonably simple.

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- When $\alpha \notin \mathbb{Q}$ we have simple real poles and we can calculate residues. $\to$ abs. conv. double series, but not for all α.

$\alpha \in (1,2)\backslash\mathbb{Q}$

Mellin inversion gives, after a considerable amount of work,

$$p(x) = x^{\alpha\rho-1} \sum_{\substack{m+\alpha(n+\frac{1}{2})<k \\ m\geq 0,\, n\geq 0}} a_{m,n} x^{m+\alpha n} + \mathrm{e}_k(x),$$

where $a_{m,n}$ are the residues of $\mathcal{M}(S_1, w)$ corresponding to the poles $s^-_{m,n}$ and

$$|\mathrm{e}_k(x)| < (A(1+x))^k \times e^{-\epsilon k \ln(k)} \times \prod_{l=1}^{k} \left| \sec\left(\frac{\pi l}{\alpha}\right)\right|.$$

Outline

Continued fractions

Recall that every real number x has a continued fraction representation

$$x = [a_0; a_1, a_2, \dots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \dots}}. \tag{1}$$

- Irrational numbers have infinite continued fraction representations.
- For irrational x truncating (1) after k steps results in a rational number $p_k/q_k(x) := [a_0; a_1, a_2, ..., a_k]$ called the k^{th} *convergent*.
- The k^{th} convergent is the best rational approximation of x among all rational numbers with denominator less than or equal to $q_k(x)$.

Continued fractions

- We have the recursive relationship $q_k = a_k q_{k-1} + q_{k-2}$, and the estimate

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}$$

Definition

The set $\mathcal{L}$ is composed of irrational numbers x for which there exists $b > 1$ such that $a_k > b^{q_k}$ for infinitely many k.

$\mathcal{L}$ is closed under addition and multiplication by rational numbers, and $x \in \mathcal{L} \Leftrightarrow x^{-1} \in \mathcal{L}$.

Back to our problem

Suppose now that $\alpha \in \mathcal{L}$ in which case so is $2/\alpha$. Then, based on our previous discussion, it it reasonable to assume $p_k/q_k(2/\alpha)$ approximates $2/\alpha$ very closely. Accordingly, we will have difficulty bounding

$$\left|\sec\left(\frac{\pi q_k}{\alpha}\right)\right| = \left|\sec\left(\frac{\pi}{2} \times q_k \times \frac{2}{\alpha}\right)\right|$$

by an exponential function of q_k. In fact, we can show quite easily that for $\alpha = [a_0; a_1, a_2, \ldots]$, $a_0 = 0$, $a_1 = 1$, and $a_{k+1} = 2^{q_k^2}$ we have

$$\left|\sec\left(\frac{\pi q_k}{\alpha}\right)\right| > \frac{2^{q_k^2}}{\pi}.$$

A solution

The key to establishing a conditionally convergent series, is to show that

$$\prod_{l=1}^{q_k - 1} \left| \sec\left(\frac{\pi l}{\alpha}\right) \right| \leq C 6^{q_k},$$

where $q_k = q_k(2/\alpha)$.

A solution

Then we replace the k with $q_k - 1$ in our previous calculation:

$$p(x) = x^{\alpha\rho - 1} \sum_{\substack{m+\alpha(n+\frac{1}{2})<q_k-1 \\ m\geq 0,\, n\geq 0}} a_{m,n} x^{m+\alpha n} + \mathrm{e}_{q_k-1}(x),$$

where we recall that

$$|\mathrm{e}_{q_k-1}(x)| < (A(1+x))^{q_k-1} \times e^{-\epsilon(q_k-1)\ln(q_k-1)} \times \prod_{l=1}^{q_k-1} \left|\sec\left(\frac{\pi l}{\alpha}\right)\right|.$$

An analogous approach works for $\alpha \in (0,1)$ which leads us to to conclude...

Main result

Theorem

There exists a conditionally convergent double series for $p(x)$ valid for all irrational α.

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