Analytical methods for Lévy processes with applications to finance, Part II

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Joint work with Alexey Kuznetsov.

1 Introduction

2 Asian options and meromorphic Lévy processes
- Theory
- Numerics and Implementation

3 Approximating Lévy processes with completely monotone jumps

Overview

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Other pricing methods

In general, pricing Asian options is difficult because they are path dependent options and $Z_t = A_0 \int_0^t e^{X_u} du$ is not a Markov process.

1. Monte Carlo simulation
2. Moment matching, Black-Scholes setting

   M.A. Milevsky and S.E. Posner. Asian options, the sum of lognormals, and the reciprocal gamma distribution. *Journal of Financial and Quantitative Analysis*, 33(3):409–422, 1998.

3. Reducing to a PDE or IDE and solving numerically:
   - The two-dimensional process (X_t, Z_t) is Markov. Derive three-dimensional PDE for C.

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   - Write C in terms of $\tilde{Z}_t := (x + Z_t)e^{-X_t}$ by a change of measure. Since $\tilde{Z}_t$ is Markov, we can compute C by solving the backward Kolmogorov equation (two-dimensional IDE).

     J. Vecer and M. Xu. Pricing Asian options in a semimartingale model. *Quantitative Finance*, 4(2):170–175, 2004.

     E. Bayraktar and H. Xing. Pricing Asian options for jump diffusions. *Mathematical Finance*, 21(1):117–143, 2011.

The distribution of $I_{\mathbf{e}(q)}$

- The hyper-exponential case (finite activity jumps)

  N. Cai and S.G. Kou. Pricing Asian options under a hyper-exponential jump diffusion model. *Operations Research*, 60(1):64–77, 2012.

- Processes with jumps of rational transform (finite activity jumps)

  A. Kuznetsov. On the distribution of exponential functionals for Lévy processes with jumps of rational transform. *Stoch. Proc. Appl.*, 122(2):654–663, 2012.

- Hyper-geometric processes (infinite activity jumps but distribution is known for only one value of q)

  A. Kuznetsov and J.C Pardo. Fluctuations of stable processes and exponential functionals of hypergeometric Lévy processes. *Acta Applicandae Mathematicae*, 123(1):113 – 139, 2013.

Asian call

Recall, we wish to compute

$$C(A_0, K, T) := e^{-rT}\mathbb{E}\left[\left(\frac{1}{T}\int_0^T A_0 e^{X_u}\mathrm{d}u - K\right)^+\right],$$

or equivalently compute

$$f(k,t) := \mathbb{E}\left[\left(\int_0^t e^{X_u}\mathrm{d}u - k\right)^+\right].$$

Asian call

Our proposed algorithm follows Cai and Kou. That is, we transform once

$$h(k,q) := q\int_{\mathbb{R}+} e^{-qt} f(k,t)\mathrm{d}t = \mathbb{E}\left[\left(\int_0^{\mathbf{e}(q)} e^{X_t}\mathrm{d}t - k\right)^+\right],$$

and then again

$$\Phi(z,q) := \int_{\mathbb{R}+} h(k,q)k^{z-1}\mathrm{d}k = \mathbb{E}\left[\int_{\mathbb{R}+} \left(I_{\mathbf{e}(q)} - k\right)^+ k^{z-1}\mathrm{d}k\right]$$
$$= \mathbb{E}\left[\int_0^{I_{\mathbf{e}(q)}} \left(I_{\mathbf{e}(q)} - k\right) k^{z-1}\mathrm{d}k\right] = \frac{\mathbb{E}\left[I_{\mathbf{e}(q)}^{z+1}\right]}{z(z+1)} = \frac{\mathcal{M}(I_{\mathbf{e}(q)}, z+2)}{z(z+1)},$$

to get an expression for the doubly transform price in terms of the Mellin transform of the exponential functional.

Products of Beta random variables

With any two unbounded sequences $\alpha = \{\alpha_n\}_{n\geq 1}$ and $\beta = \{\beta_n\}_{n\geq 1}$ which satisfy the interlacing property

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 \ \ldots$$

we associate an infinite product of independent beta random variables, defined as

$$J(\alpha, \beta) := \prod_{n\geq 1} B_{(\alpha_n,\, \beta_n - \alpha_n)} \frac{\beta_n}{\alpha_n}.$$

Lemma

$J(\alpha, \beta)$ converges a.s.

Main Result

Theorem (H. and Kuznetsov, 2014)

Assume that $q > 0$. Define $\hat{\rho}_0 := 0$ and the four sequences

$\zeta := \{\zeta_n\}_{n\geq 1}$, $\rho := \{\rho_n\}_{n\geq 1}$, $\tilde{\zeta} := \{1 + \hat{\zeta}_n\}_{n\geq 1}$, $\tilde{\rho} := \{1 + \hat{\rho}_{n-1}\}_{n\geq 1}$.

Then we have the following identity in distribution

$$I_{\mathbf{e}(q)} \stackrel{d}{=} C(q) \times \frac{J(\tilde{\rho}, \tilde{\zeta})}{J(\zeta, \rho)},$$

where $C(q)$ is a constant and the random variables $J(\tilde{\rho}, \tilde{\zeta})$ and $J(\zeta, \rho)$ are independent. cont. $\rightarrow$

Main Result

Theorem (cont.)

The Mellin transform $\mathcal{M}(I_{\mathbf{e}(q)}, z)$ is finite for $0 < \mathrm{Re}(z) < 1 + \zeta_1$ and is given by

$$\mathcal{M}(I_{\mathbf{e}(q)}, z) = C^{z-1} \overbrace{\prod_{n\geq 1} \frac{\Gamma(\hat{\zeta}_n + 1)\Gamma(\hat{\rho}_{n-1} + z)}{\Gamma(\hat{\rho}_{n-1} + 1)\Gamma(\hat{\zeta}_n + z)} \left(\frac{\hat{\zeta}_n + 1}{\hat{\rho}_{n-1} + 1}\right)^{z-1}}^{\mathcal{M}(J(\tilde{\rho},\tilde{\zeta}),z)} \times \underbrace{\prod_{n\geq 1} \frac{\Gamma(\rho_n)\Gamma(\zeta_n + 1 - z)}{\Gamma(\zeta_n)\Gamma(\rho_n + 1 - z)} \left(\frac{\zeta_n}{\rho_n}\right)^{z-1}}_{\mathcal{M}(J(\zeta,\rho),2-z)}.$$

D. Hackmann and A. Kuznetsov.
Asian options and meromorphic Lévy processes.
Finance and Stochastics, 18:825–844, 2014.

A rough idea of the proof

We use the verification result of Kuznetsov and Pardo: A function $f(z)$ is the Mellin transform of $I_{\mathbf{e}(q)}$ if

1. for some $\theta > 0$, the function $f(z)$ is analytic and zero free in the vertical strip $0 < \mathrm{Re}(z) < 1 + \theta$;
2. the function $f(z)$ satisfies

$$f(z+1) = \frac{z}{q - \psi(z)} f(z), \quad 0 < z < \theta,$$

   where $\psi(z)$ is the Laplace exponent of the process X;
3. $|f(z)|^{-1} = o(\exp(2\pi|\mathrm{Im}(z)|))$ as $\mathrm{Im}(z) \to \infty$, uniformly in the strip $0 < \mathrm{Re}(z) < 1 + \theta$.

A rough idea of the proof

We need to find a candidate function $f(z)$ and we let point 2 guide us. We are aided by the fact that $q-\psi(z)$ is just a product of linear factors involving the roots and poles. That is,

$$q-\psi(z) = q\prod_{n\geq 1}\frac{1-\frac{z}{\zeta_n}}{1-\frac{z}{\rho_n}} \times \prod_{n\geq 1}\frac{1+\frac{z}{\hat{\zeta}_n}}{1+\frac{z}{\hat{\rho}_n}}, \quad z\in\mathbb{C},$$

where the two infinite products converge.

A. Kuznetsov. Wiener-Hopf factorization for a family of Lévy processes related to theta functions. *Journal of Applied Probability*, 47(4):1023–1033, 2010.

A rough idea of the proof

Therefore, we are solving many simpler functional equations of the form:

$$f(z+1) = (a \pm z)^k f(z),$$

where a represents a root or a pole, and $k \in \{-1, 1\}$. A solution of such an equation can readily be obtained using the well known formula

$$\Gamma(z+1) = z\Gamma(z),$$

for the gamma function.

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Setup

To obtain the price we need to compute $h(k,q)$ as the inverse Mellin transform

$$h(k,q) = \frac{k^{-d_1}}{2\pi} \int_{\mathbb{R}} \frac{\mathcal{M}(I_{\mathbf{e}(q)}, d_1 + iv + 2)}{(d_1 + iv)(d_1 + iv + 1)} e^{-iv \log(k)} \mathrm{d}v,$$

where $d_1 \in (0, \zeta_1(d_2) - 1)$, $q = d_2 + iu$, and $d_2 > r$. Second, we compute $f(k,t)$ as the inverse Laplace transform, which can be rewritten as the cosine transform

$$f(k,t) = \frac{2e^{d_2 t}}{\pi} \int_{\mathbb{R}^+} \mathrm{Re}\left(\frac{h(k, d_2 + iu)}{d_2 + iu}\right) \cos(ut) \mathrm{d}u.$$

Implementation

The steps we need to follow/hurdles we need to overcome are:

- Choose a process

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- Choose a process
- Evaluate $\mathcal{M}(I_{\mathbf{e}(q)}, z)$ for complex q
- Truncate $\mathcal{M}(I_{\mathbf{e}(q)}, z)$ efficiently

The process

We will use a theta process for which we have a closed form formula for $\psi(z)$. We can manipulate parameters of the the process to give a process with infinite activity and variation.

Parameter Set I will give a process with a Gaussian component and jumps of infinite activity but finite variation.

Parameter Set II gives a process with zero Gaussian component and jumps of infinite variation.

Complex q

Unfortunately, we do not know whether or formula for $\mathcal{M}(I_{\mathbf{e}(q)}, z)$ is valid for complex q. Our numerical experiments support the conjecture that it is.

What about finding the roots $\{\zeta_n, -\hat{\zeta}_n\}_{n\geq 1}$ when $q = q_0 + iu$, $u \in \mathbb{R}^+$?

Complex q

We may view $\zeta_n(u)$ as an implicitly defined function of u which satisfies,

$$q_0 + iu - \psi(\zeta_n(u)) = 0, \quad \zeta_n(0) = \zeta_n,$$

where ζ_n is the solution of $\psi(z) = q_0$. Differentiating each side with respect to u gives the ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}u}\zeta_n(u) = \frac{i}{\psi'(\zeta_n(u))},$$

with initial condition $\zeta_n(0) = \zeta_n$. Such an equation can be solved nicely by a numerical scheme like the midpoint method.

Truncating $\mathcal{M}(I_{\mathbf{e}(q)}, z)$

To approximate $\mathcal{M}(I_{\mathbf{e}(q)}, z)$ we can simply truncate our infinite product, but convergence may be slow. The more terms we need, the more roots $\{\zeta_n, \hat{\zeta}_n\}_{n\geq 1}$ we need to calculate which is computationally expensive. Note if we truncate the transform we get:

$$\mathcal{M}_N(z) := a_N \times b_N^{z-1} \times \prod_{n=1}^{N} \frac{\Gamma(\hat{\rho}_{n-1} + z)}{\Gamma(\hat{\zeta}_n + z)} \frac{\Gamma(\zeta_n + 1 - z)}{\Gamma(\rho_n + 1 - z)}$$

where and a_N and b_N are normalizing constants.

Truncating $\mathcal{M}(I_{\mathbf{e}(q)}, z)$

Now we note that

$$\mathcal{M}(I_{\mathbf{e}(q)}, z) = \mathcal{M}_N(z) R_N(z)$$

where $R_N(z) = \mathcal{M}(I_{\mathbf{e}(q)}, z)/\mathcal{M}_N(z)$ is the Mellin transform of the tail of our product of beta random variables which we denote $\epsilon^{(N)}$. Instead of simply letting $R_N(s) = 1$ we try to find a random variable ξ matching the moments of $\epsilon^{(N)}$.

Truncating $\mathcal{M}(I_{\mathbf{e}(q)}, z)$

We can calculate the moments m_k using the functional equation $\mathcal{M}(I_{\mathbf{e}(q)}, z+1) = z\mathcal{M}(I_{\mathbf{e}(q)}, z)/(q - \psi(z))$, we find

$$m_k = R_N(k+1) = \frac{\mathcal{M}(I_{\mathbf{e}(q)}, k+1)}{\mathcal{M}_N(k+1)} = \frac{k!}{\mathcal{M}_N(k+1)} \prod_{j=1}^{k} \frac{1}{q - \psi(j)}.$$

Truncating $\mathcal{M}(I_{\mathbf{e}(q)}, z)$

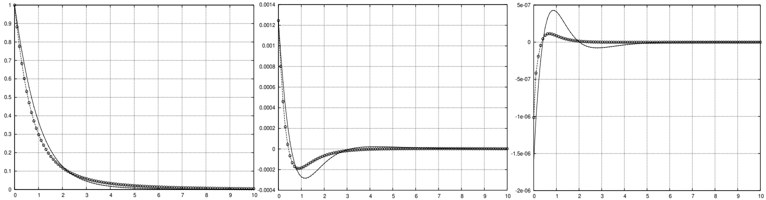
Finally we let ξ be a beta random variable of the second kind which has density:

$$\mathbb{P}(\xi \in \mathrm{d}x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} y^{a-1}(1+y)^{-a-b}\mathrm{d}y, \quad y > 0.$$

We choose $a, b > 0$ such $\mathbb{E}[\xi] = m_1$ and $\mathbb{E}[\xi^2] = m_2$, and replace $R_N(z)$ with the Mellin transform of ξ which has the form:

$$\mathbb{E}[\xi^{z-1}] = \frac{\Gamma(a+z-1)\Gamma(b+1-z)}{\Gamma(a)\Gamma(b)}.$$

A test: Calculating the density of $I_{\mathbf{e}(1)}$



(a) (b) (c)

Figure : (a) The density of the exponential functional $I_{\mathbf{e}(1)}$ with $N = 400$ (the benchmark). (b) The error with $N = 20$ (no correction). (c) The error with $N = 20$ (with correction term). Solid line (resp. circles) represent parameter set I (resp. II).

Numerics: Pricing an Asian Option Results

N	Algorithm 1, price	Time (sec.)	Algorithm 2, price	Time (sec.)
10	4.724627	1.6	4.720675	1.2
20	4.727780	2.8	4.728032	1.8
40	4.728013	4.8	4.728031	3.4
80	4.728029	9.2	4.728031	7.1

Table : The price of the Asian option, parameter set I. The Monte-Carlo estimate of the price is 4.7386 with the standard deviation 0.0172. The exact price is 4.72802±1.0e-5.

Option parameters: $A_0 = 100$, $T = 1$, $K = 105$, and $r = 0.03$, with risk neutral condition $\psi(1) = r$ satisfied (this and the assumption $\rho_1 > 1$ ensures key quantities are finite).

Numerics: Pricing an Asian Option Results

N	Algorithm 1, price	Time (sec.)	Algorithm 2, price	Time (sec.)
10	10.620243	1.6	10.621039	1.2
20	10.620049	3.0	10.620171	2.2
40	10.620037	4.8	10.620054	3.6
80	10.620036	9.6	10.620039	7.4

Table : The price of the Asian option, parameter set II. The Monte-Carlo estimate of the price is 10.6136 with the standard deviation 0.0251. The exact price is 10.62003±1.0e-5.

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Classification: Completely monotone jumps

Definition

A function $f(x)$ is called completely monotone if $(-1)^n f^{(n)}(x) > 0$ for all $x > 0$, $n = 0, 1, 2, \ldots$.

Definition

A Lévy process has completely monotone jumps, if the Lévy measure is absolutely continuous with density $\pi(x)$, and $\pi(x)$ and $\pi(-x)$ are completely monotone for $x \in (0, \infty)$.

Assumption: From now on we assume all processes have completely monotone jumps and $\pi(x)$ decreases exponentially fast as $x \to \pm\infty$.

Some facts

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- The jump density of a process X is completely monotone if, and only if, S_q and I_q are mixtures of exponentials.

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L.C.G. Rogers.
Weiner-Hopf factorization of diffusions and Lévy processes.
Proc. Lond. Math. Soc., 47(3):177–191, 1983.

Main idea

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- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$.
- The Laplace exponent of a hyper-exponential process is a rational function.
- Thus we have two problems:

  (1) Approximate $\psi(z)$ by a rational function $\tilde{\psi}(z)$,

  (2) Show that $\tilde{\psi}(z)$ is itself a Laplace exponent of a Lévy process.

Padé approximation

Definition

Let f be a function with a power series representation $f(z) = \sum_{i=0}^{\infty} c_i z^i$. If there exist polynomials $P_m(z)$ and $Q_n(z)$ satisfying $\deg(P) \leq m$, $\deg(Q) \leq n$, $Q_n(0) = 1$ and

$$\frac{P_m(z)}{Q_n(z)} = c_0 + c_1 z + \cdots + c_{m+n} z^{m+n} + O(z^{m+n+1}), \quad z \to 0,$$

then we say that $f^{[m/n]}(z) := P_m(z)/Q_n(z)$ is the $[m/n]$ *Padé approximant of* f.

A simple example of Padé approximations

m \ n	0	1	2	3
0	$\frac{1}{1}$	$\frac{1}{1-z}$	$\frac{1}{1-z+\frac{1}{2}z^2}$	$\frac{1}{1-z+\frac{1}{2}z^2-\frac{1}{6}z^3}$
1	$\frac{1+z}{1}$	$\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}$	$\frac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}$	$\frac{1+\frac{1}{4}z}{1-\frac{3}{4}z+\frac{1}{4}z^2-\frac{1}{24}z^3}$
2	$\frac{1+z+\frac{1}{2}z^2}{1}$	$\frac{1+\frac{2}{3}z+\frac{1}{6}z^2}{1-\frac{1}{3}z}$	$\frac{1+\frac{1}{2}z+\frac{1}{12}z^2}{1-\frac{1}{2}z+\frac{1}{12}z^2}$	$\frac{1+\frac{2}{5}z+\frac{1}{20}z^2}{1-\frac{3}{5}z+\frac{3}{20}z^2-\frac{1}{60}z^3}$
3	$\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3}{1}$	$\frac{1+\frac{3}{4}z+\frac{1}{4}z^2+\frac{1}{24}z^3}{1-\frac{1}{4}z}$	$\frac{1+\frac{3}{5}z+\frac{3}{20}z^2+\frac{1}{60}z^3}{1-\frac{2}{5}z+\frac{1}{20}z^2}$	$\frac{1+\frac{1}{2}z+\frac{1}{10}z^2+\frac{1}{120}z^3}{1-\frac{1}{2}z+\frac{1}{10}z^2-\frac{1}{120}z^3}$
4	$\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4}{1}$	$\frac{1+\frac{4}{5}z+\frac{3}{10}z^2+\frac{1}{15}z^3+\frac{1}{120}z^4}{1-\frac{1}{5}z}$	$\frac{1+\frac{2}{3}z+\frac{1}{5}z^2+\frac{1}{30}z^3+\frac{1}{360}z^4}{1-\frac{1}{3}z+\frac{1}{30}z^2}$	$\frac{1+\frac{4}{7}z+\frac{1}{7}z^2+\frac{2}{105}z^3+\frac{1}{840}z^4}{1-\frac{3}{7}z+\frac{1}{14}z^2-\frac{1}{210}z^3}$

Figure : The initial part of the Padé table for e^z

Gaussian quadrature

- ν is a finite positive measure on a closed bounded interval $[a, b]$

Gaussian quadrature

- ν is a finite positive measure on a closed bounded interval $[a, b]$
- For each n we want to find a measure $\tilde{\nu}_n$ on a finite number of points in $[a, b]$ such that we match the first $2n - 1$ moments of ν, i.e.

$$\int_{[a,b]} x^j \nu(\mathrm{d}x) = \sum_i^n x_i^j w_i, \quad ,\text{for } j = 1, \ldots, 2n - 1.$$

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$$\int_{[a,b]} x^j \nu(\mathrm{d}x) = \sum_i^n x_i^j w_i, \quad \text{, for } j = 1, \ldots, 2n-1.$$

- The points x_i and w_i are the nodes and weights of the Gaussian quadrature.

Gaussian quadrature and orthogonal polynomials

- $\{p_n(x)\}_{n\geq 0}$ be the sequence of orthogonal polynomials with respect to the measure $\nu(\mathrm{d}x)$: $\deg(p_n) = n$ and

$$(p_n, p_m)_\nu := \int_{[a,b]} p_n(x)p_m(x)\nu(dx) = d_n\delta_{n,m}$$

G. Szegö.
Orthogonal Polynomials.
Amer. Math. Soc., Providence, RI, 4 edition, 1975.

Gaussian quadrature and orthogonal polynomials

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$$(p_n, p_m)_\nu := \int_{[a,b]} p_n(x) p_m(x) \nu(dx) = d_n \delta_{n,m}$$

- The nodes of the Gaussian quadrature $\tilde{\nu}_n$ are the zeros of p_n and the weights may be calculated from p_{n-1}, p_n.

G. Szegö.
Orthogonal Polynomials.
Amer. Math. Soc., Providence, RI, 4 edition, 1975.

Bernstein's Theorem

We can develop a very useful description of the processes which satisfy our assumption using Bernstein's theorem. A process satisfies our assumption if, and only if, there exists a positive measure $\mu(\mathrm{d}u)$, with support in $\mathbb{R}\backslash\{0\}$, such that for all $x \in \mathbb{R}$

$$\pi(x) = \mathbb{I}(x > 0) \int_{(0,\infty)} e^{-ux} \mu(\mathrm{d}u) + \mathbb{I}(x < 0) \int_{(-\infty,0)} e^{-ux} \mu(\mathrm{d}u), \quad (1)$$

and $\mu(\mathrm{d}u)$ assigns no mass to a non-empty interval $(-\hat{\rho}, \rho)$ containing the origin + integrability condition on $\mu(\mathrm{d}u)$.

A change of variables

We define

$$\mu^*(A) := \mu(\{v \in \mathbb{R} \ : \ v^{-1} \in A\}).$$

Then, the Lévy-Khintchine formula + Fubini's theorem + change of variables give us

$$\psi(z) = \frac{\sigma^2}{2} z^2 + az + z^2 \int_{[-\hat{\rho}^{-1}, \rho^{-1}]} \frac{|v|^3 \mu^*(\mathrm{d}v)}{1 - vz}.$$

Key Observation: $|v|^3 \mu^*(\mathrm{d}v)$ is a finite measure, with bounded support.

Main theorem (two-sided case)

Assume that $\sigma = 0$. Let $\{x_i\}_{1\le i\le n}$ and $\{w_i\}_{1\le i\le n}$ be the nodes and the weights of the Gaussian quadrature of order n with respect to the measure $|v|^3\mu^*(\mathrm{d}v)$. We define

$$\psi_n(z) := az + z^2 \sum_{i=1}^{n} \frac{w_i}{1 - zx_i}.$$

Theorem (H. and Kuznetsov, 2014)

(i) *The function $\psi_n(z)$ is the $[n+1/n]$ Padé approximant of $\psi(z)$.*

(ii) *The function $\psi_n(z)$ is the Laplace exponent of a hyper-exponential process $X^{(n)}$ having the characteristic triple $(a, \sigma_n^2, \pi_n)_{h\equiv x}$, where*

Main theorem (two-sided case)

Theorem (cont.)

(ii)

$$\pi_n(x) := \begin{cases} \sum_{1\le i\le n\,:\,x_i<0} w_i|x_i|^{-3}e^{-\frac{x}{x_i}}, & \text{if } x<0, \\ \sum_{1\le i\le n\,:\,x_i>0} w_i x_i^{-3}e^{-\frac{x}{x_i}}, & \text{if } x>0. \end{cases}$$

(iii) *The random variables* $X_1^{(n)}$ *and* X_1 *satisfy* $\mathbb{E}[(X_1^{(n)})^j] = \mathbb{E}[(X_1)^j]$ *for* $1 \le j \le 2n+1$.

Convergence

Theorem (H. and Kuznetsov, 2014)

For any compact set $A \subset \mathbb{C} \setminus \{(-\infty, -\hat{\rho}] \cup [\rho, \infty)\}$ *there exist* $c_1 = c_1(A) > 0$ *and* $c_2 = c_2(A) > 0$ *such that for all* $z \in A$ *and all* $n \geq 1$

$$|\psi_n(z) - \psi(z)| < c_1 e^{-c_2 n}.$$

D. Hackmann and A. Kuznetsov.
Approximating Lévy processes with completely monotone jumps.
http://arxiv.org/abs/1404.0597, 2014.
Forthcoming in The Annals of Applied Probability.

One-sided processes

- For CM subordinators, all three functions $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyper-exponential processes.

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One-sided processes

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- For CM spectrally-positive processes of infinite variation, only two functions $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyper-exponential processes.
- There exist explicit formulas for a number of important examples: In the VG case we have $\psi^{[n/n]}(z) = P_n(z)/Q_n(z)$, where

$$P_n(z) = 2\sum_{j=0}^{n} \binom{n}{j}^2 \left[H_{n-j} - H_j\right](1-z)^j, Q_n(z) = z^n P_n\left(\tfrac{2}{z} - 1\right).$$

and $H_j := 1 + 1/2 + \cdots + 1/j$.

How do we prove all these results?

- One can show that only $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$ and $\psi^{[n+2/n]}(z)$ can possibly be Laplace exponents of a Lévy process

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- One can show that only $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$ and $\psi^{[n+2/n]}(z)$ can possibly be Laplace exponents of a Lévy process
- The function

$$g(z) = \int_{[-\hat{\rho}^{-1},\rho^{-1}]} \frac{|v|^3 \mu^*(\mathrm{d}v)}{1 - vz}.$$

is closely related to a *Stieltjes function*:

$$f(z) := \int_{[0,R^{-1}]} \frac{\nu(\mathrm{d}u)}{1 + zu}$$

Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n - 1$.

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$$f^{[n-1/n]}(z) = \frac{(-z)^{n-1} q_{n-1}(-1/z)}{(-z)^n p_n(-1/z)} = \sum_{i=1}^{n} \frac{w_i}{1 + x_i z}.$$

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G.D. Allen, C.K. Chui, W.R. Madych, F.J. Narcowich, and P.W. Smith.
Padé approximation of Stieltjes series.
Journal of approximation theory, 14:302–316, 1975.

Math Finance applications

We will work with the following two processes: the VG process V defined by the Laplace exponent

$$\psi(z) = \mu z - c \log\left(1 - \frac{z}{\rho}\right) - c \log\left(1 + \frac{z}{\hat{\rho}}\right),$$

and parameters

$$(\rho, \hat{\rho}, c) = (21.8735,\ 56.4414,\ 5.0),$$

and the CGMY process Z defined by the Laplace exponent

$$\psi(z) = \mu z + C\Gamma(-Y)\left[(M - z)^Y - M^Y + (G + z)^Y - G^Y\right],$$

and parameters

$$(C, G, M, Y) = (1,\ 8.8,\ 14.5,\ 1.2).$$

A test: European call

	two-sided $[2N+1/2N]$	one-sided $[N+1/N]$	one-sided $[N+2/N]$
$N=1$	-2.75e-2	1.93e-2	-3.72e-3
$N=2$	-4.86e-6	-4.19e-6	9.5e-5
$N=3$	4.80e-7	-1.48e-5	-2.54e-7
$N=4$	2.9e-8	6.41e-7	-1.55e-7
$N=5$	1.14e-9	5.58e-9	6.95e-9

Table : The error in computing the price of the European call option for the CGMY Z-model. Initial stock price is $A_0 = 100$, strike price $K = 100$, maturity $T = 0.25$ and interest rate $r = 0.04$. The benchmark price is 11.9207826467.

Down-and-out put

$$p_I(y) = \sum_{i=1}^{\hat{N}+1} \hat{c}_i \hat{\zeta}_i e^{\hat{\zeta}_i y}, \quad y < 0, \quad \text{and} \quad p_S(x) = \sum_{j=1}^{N+1} c_j \zeta_j e^{-\zeta_j x}, \quad x > 0.$$

$$\begin{aligned}
F(q) &= \mathbb{E}[(k - e^{S_q + I_q})^+ \mathbb{I}(I_q > b)] \\
&= \int_0^{-b} \int_0^{\log(k)+y} (k - e^{x-y}) p_S(x) p_I(-y) \mathrm{d}x\mathrm{d}y \\
&= \sum_{i=1}^{\hat{N}+1} \sum_{j=1}^{N+1} \frac{\hat{c}_i c_j}{\zeta_j - 1} \times \\
&\quad \left(k(e^{b\hat{\zeta}_i} - 1)(1 - \zeta_j) - \frac{k^{1-\zeta_j} \hat{\zeta}_i (e^{b(\zeta_j + \hat{\zeta}_i)} - 1)}{\hat{\zeta}_i + \zeta_j} + \frac{\hat{\zeta}_i \zeta_j (e^{b(1+\hat{\zeta}_i)} - 1)}{\hat{\zeta}_i + 1} \right).
\end{aligned}$$

Down-and-out put

We calculate barrier option prices for the process V, for four values $A_0 \in \{81, 91, 101, 111\}$ and with other parameters given by $K = 100$, $B = 80$, $r = 0.04879$ and $T = 0.5$

	$A_0 = 81$	$A_0 = 91$	$A_0 = 101$	$A_0 = 111$
Benchmark	3.39880	7.38668	1.40351	0.04280
$N = 2$	3.44551	7.39225	1.40527	0.04233
$N = 4$	3.40209	7.38957	1.40329	0.04258
$N = 6$	3.39910	7.38939	1.40332	0.04258
$N = 8$	3.39856	7.38936	1.40332	0.04258
$N = 10$	3.39853	7.38936	1.40332	0.04258

Table : Barrier option prices calculated for the VG process V-model. Benchmark prices obtained from Kudryavtsev and Levendorskiĭ 2009

O. Kudryavtsev and S. Levendorskiĭ.
Fast and accurate pricing of barrier options under Lévy processes.
Finance Stoch., 13:531–562, 2009.