Asian options and meropmorphic Lévy processes

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Outline

Introduction

We are interested in continuously sampled arithmetic rate options with fixed maturity. The price under risk neutral measure for this type of (call) option can then be represented by

$$C(S_0, K, T) := e^{-rT}\mathbb{E}\left[\left(S_0 \int_0^T e^{X_u}\mathrm{d}u - K\right)^+\right].$$

Using the approach pioneered by Cai and Kou 2010 for hyper-exponential processes we exploit a connection with the exponential functional of the process X given by

$$I_q := \int_0^{\mathrm{e}(q)} e^{X_u}\mathrm{d}u,$$

which is an object that is the topic of much recent research. See for example Bertoin and Yor 2005.

Outline

Existing pricing approaches

The complication is that we have a path dependent option, so that we no longer necessarily have a Markov process for $Z_t = S_0 \int_0^t e^{X_u} \mathrm{d}u$.

1. Monte Carlo simulation
2. Moment matching. GBM, Milevsky and Posner 1998.
3. Reducing to a PDE or IDE and solving numerically:
   - The two-dimensional process (X_t, Z_t) is Markov. Derive three-dimensional PDE for C. GBM, Shreve, 2004.
   - $\tilde{Z}_t := (x + Z_t)e^{-X_t}$ is a Markov process. Via change of measure, we can write the price of the option in terms of $\tilde{Z}_t$ only and compute by solving the backward Kolmogorov equation (two-dimensional IDE). Semimartingales, Vecer and Xu, 2004, jump diffusion models, Bayraktar and Xing, 2011.

Existing pricing approaches cont.

4. Find the Mellin transform of $I_q := \int_0^{e(q)} e^{X_u} du$ and invert twice to find the price. Cai and Kou, 2010 for GBM and hyper-exponential Lévy processes.

We have not seen any papers which price these options in the general setting of processes with jumps of infinite activity and infinite variation other than with Monte Carlo methods.

The distribution of I_q

- Kai and Cou, 2010 for the hyper-exponential case (finite activity jumps)
- Kuznetsov 2012 for processes of jumps of rational transform (finite activity jumps)
- A. Kuznetsov and J.C. Pardo, 2013 for hyper-geometric processes (infinite activity jumps but distribution is known for only one value of q)

Outline

Motivation

We notice that $C(S_0, K, T) = \exp(-rT) \times S_0 \times f(K/S_0, T)$ where

$$f(k,t) := \mathbb{E}\left[\left(\int_0^t e^{X_u} \mathrm{d}u - k\right)^+\right],$$

and that

$$h(k,q) = \mathbb{E}[(I_q - k)^+] = \mathbb{E}[f(k, \mathrm{e}(q))] = q\int_0^\infty e^{-qt} f(k,t)\mathrm{d}t.$$

Motivation cont.

Now,

$$\begin{aligned}\int_0^\infty h(k,q)k^{s-1}\mathrm{d}k &= \mathbb{E}\left[\int_0^\infty (I_q - k)^+ k^{s-1}\mathrm{d}k\right] \\ &= \mathbb{E}\left[\int_0^{I_q} (I_q - k)\, k^{s-1}\mathrm{d}k\right] \\ &= \frac{\mathbb{E}\left[I_q^{s+1}\right]}{s(s+1)} = \frac{\mathcal{M}(s+2,q)}{s(s+1)},\end{aligned}$$

where $\mathcal{M}(s) = \mathbb{E}[I_q^{s-1}]$.

Motivation cont.

Therefore, if we can calculate (or approximate) $\mathcal{M}(s)$ we can find $h(k,q)$ via Mellin transform inversion,

$$h(k,q) = \frac{k^{-d_1}}{2\pi} \int_{\mathbb{R}} \frac{\mathcal{M}(d_1 + \mathrm{i}v + 2, q)}{(d_1 + \mathrm{i}v)(d_1 + \mathrm{i}v + 1)} e^{-\mathrm{i}v \ln(k)} \mathrm{d}v,$$

for a properly chosen d_1. Then we can recover $f(k,t)$ via the inverse Laplace transform,

$$f(k,t) = \frac{2e^{d_2 t}}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{h(k, d_2 + \mathrm{i}u)}{d_2 + \mathrm{i}u}\right] \cos(ut) \mathrm{d}u,$$

for properly chosen d_2. This will give us the price.

Outline

Meromorphic Lévy processes

A Lévy processes with Lévy measure having density

$$\pi(x) = \mathbb{I}_{\{x>0\}} \sum_{n\geq 1} a_n \rho_n e^{-\rho_n x} + \mathbb{I}_{\{x<0\}} \sum_{n\geq 1} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x}.$$

All the coefficients a_n, $\hat{a}_n$, ρ_n and $\hat{\rho}_n$ are strictly positive and the sequences $\{\rho_n\}_{n\geq 1}$ and $\{\hat{\rho}_n\}_{n\geq 1}$ must be strictly increasing and unbounded.

Meromorphic Lévy processes cont.

The Laplace exponent $\psi(z) := \ln \mathbb{E}[\exp(zX_1)]$ of the meromorphic process X is given for , $-\hat{\rho}_1 < \mathrm{Re}(z) < \rho_1$, by

$$\psi(z) = \frac{1}{2}\sigma^2 z^2 + \mu z + z^2 \sum_{n\geq 1} \frac{a_n}{\rho_n(\rho_n - z)} + z^2 \sum_{n\geq 1} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)}.$$

By analytic continuation we see that ψ is a meromorphic function on $\mathbb{C}$. Importantly for any $q > 0$ the equation $\psi(z) = q$ has only simple real solutions ζ_n, $-\hat{\zeta}_n$, that satisfy

$$\ldots - \hat{\rho}_2 < -\hat{\zeta}_2 < -\hat{\rho}_1 < -\hat{\zeta}_1 < 0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \ldots$$

Outline

Introduction

Existing pricing approaches and the exponential functional I_q

Methodology overview

Meromorphic Lévy processes

Theoretical results

Numerical results

Products of Beta random variables

With any two unbounded sequences $\alpha = \{\alpha_n\}_{n\geq 1}$ and $\beta = \{\beta_n\}_{n\geq 1}$ which satisfy the interlacing property

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 \ \ldots$$

we associate an infinite product of independent beta random variables, defined as

$$X(\alpha, \beta) := \prod_{n\geq 1} B_{(\alpha_n,\ \beta_n - \alpha_n)} \frac{\beta_n}{\alpha_n}.$$

Lemma

$X(\alpha, \beta)$ converges a.s.

Proof

Main Result

Theorem

Assume that $q > 0$. Define $\hat{\rho}_0 = 0$ and the four sequences

$$\zeta = \{\zeta_n\}_{n\geq 1},\ \rho = \{\rho_n\}_{n\geq 1},\ \tilde{\zeta} = \{1 + \hat{\zeta}_n\}_{n\geq 1},\ \tilde{\rho} = \{1 + \hat{\rho}_{n-1}\}_{n\geq 1}.$$

Then we have the following identity in distribution

$$I_q \stackrel{d}{=} C(q) \times \frac{X(\tilde{\rho}, \tilde{\zeta})}{X(\zeta, \rho)},$$

where $C(q)$ is a constant and the random variables $X(\tilde{\rho}, \tilde{\zeta})$ and $X(\zeta, \rho)$ are independent. cont. $\rightarrow$

Main Result cont.

Theorem cont.

The Mellin transform $\mathcal{M}(s) = \mathcal{M}(s,q) := \mathbb{E}[I_q^{s-1}]$ is finite for $0 < \mathrm{Re}(s) < 1 + \zeta_1$ and is given by

$$\mathcal{M}(s) = C^{s-1} \overbrace{\prod_{n\geq 1} \frac{\Gamma(\hat{\zeta}_n + 1)\Gamma(\hat{\rho}_{n-1} + s)}{\Gamma(\hat{\rho}_{n-1} + 1)\Gamma(\hat{\zeta}_n + s)} \left(\frac{\hat{\zeta}_n + 1}{\hat{\rho}_{n-1} + 1}\right)^{s-1}}^{M_1(s)} \times \underbrace{\prod_{n\geq 1} \frac{\Gamma(\rho_n)\Gamma(\zeta_n + 1 - s)}{\Gamma(\zeta_n)\Gamma(\rho_n + 1 - s)} \left(\frac{\zeta_n}{\rho_n}\right)^{s-1}}_{M_2(2-s)}.$$

Proof

Outline

Numerics

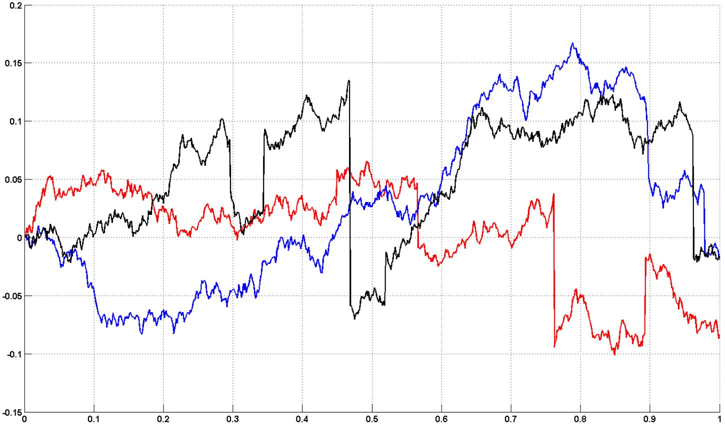
We will use a theta process for which we have a closed form formula for ψ. We can manipulate parameters of the the process to give a process with infinite activity and variation.

Parameter Set I will give a process with a Gaussian component and jumps of infinite activity but finite variation.

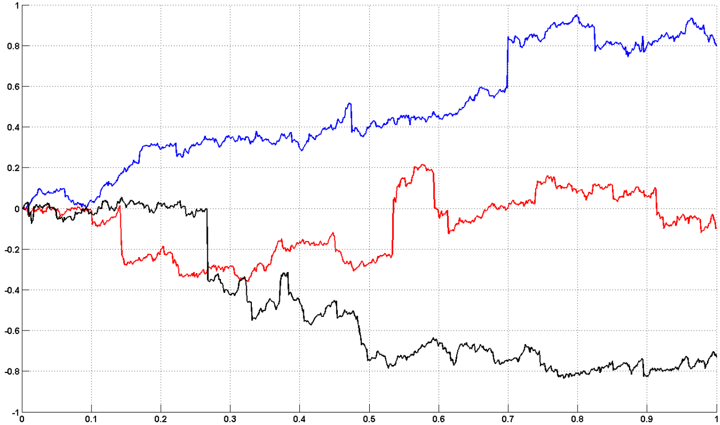
Parameter Set II gives a process with zero Gaussian component and jumps of infinite variation.

$S_0 = 100$, $T = 1$, $K = 105$, and $r = 0.03$, with risk neutral condition $\psi(1) = r$ satisfied.

Numerics



Numerics



Numerics: Truncating $\mathcal{M}(s)$

To approximate $\mathcal{M}(s)$ we can simply truncate our infinite product, but convergence may be slow. The more terms we need, the more roots $\tilde{\zeta}$ and ζ we need to calculate which is computationally expensive. Note if we truncate the transform we get:

$$\mathcal{M}_N(s) := a_N \times b_N^{s-1} \times \prod_{n=1}^{N} \frac{\Gamma(\hat{\rho}_{n-1} + s)}{\Gamma(\hat{\zeta}_n + s)} \frac{\Gamma(\zeta_n + 1 - s)}{\Gamma(\rho_n + 1 - s)}$$

where and a_N and b_N are normalizing constants.

Numerics: Truncating $\mathcal{M}(s)$ cont.

Now we note that

$$\mathcal{M}(s) = \mathcal{M}_N(s) R_N(s)$$

where $R_N(S) = \mathcal{M}(s)/\mathcal{M}_N(s)$ is the Mellin transform of the tail of our product of beta random variables which we denote $\epsilon^{(N)}$. Instead of simply letting $R_N(s) = 1$ we try to find a random variable ξ matching the first two moments m_1 and m_2 of $\epsilon^{(N)}$.

More

Numerics: Truncating $\mathcal{M}(s)$ cont.

We let ξ be a beta random variable of the second kind which has density:

$$\mathbb{P}(\xi \in \mathrm{d}x) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} y^{a-1}(1+y)^{-a-b}\mathrm{d}y, \quad y > 0.$$

We choose $a, b > 0$ such $\mathbb{E}[\xi] = m_1$ and $\mathbb{E}[\xi^2] = m_2$.

Numerics: Pricing an Asian Option Results

N	Algorithm 1, price	Time (sec.)	Algorithm 2, price	Time (sec.)
10	4.724627	1.6	4.720675	1.2
20	4.727780	2.8	4.728032	1.8
40	4.728013	4.8	4.728031	3.4
80	4.728029	9.2	4.728031	7.1

Table: The price of the Asian option, parameter set I. The Monte-Carlo estimate of the price is 4.7386 with the standard deviation 0.0172. The exact price is 4.72802±1.0e-5.

Details

Numerics: Pricing an Asian Option Results

N	Algorithm 1, price	Time (sec.)	Algorithm 2, price	Time (sec.)
10	10.620243	1.6	10.621039	1.2
20	10.620049	3.0	10.620171	2.2
40	10.620037	4.8	10.620054	3.6
80	10.620036	9.6	10.620039	7.4

Table: The price of the Asian option, parameter set II. The Monte-Carlo estimate of the price is 10.6136 with the standard deviation 0.0251. The exact price is 10.62003$\pm$1.0e-5.

References

E. Bayraktar and H. Xing.
Pricing Asian options for jump diffusions.
Mathematical Finance, 21(1):117–143, 2011.

J. Bertoin and M. Yor.
Exponential functionals of Lévy processes.
Probab. Surv., 2:191–212 (electronic), 2005.

N. Cai and S.G. Kou.
Pricing Asian options under a general jump diffusion model.
Operations Research, 60(1):64–77, 2012.

D. Hackmann and A. Kuznetsov.
Asian options and meromorphic Lévy processes.
http://arxiv.org/abs/1305.0725v1, 2013.
Preprint.

M.A. Milevsky and S.E. Posner.
Asian options, the sum of lognormals, and the reciprocal gamma distribution.
Journal of Financial and Quantitative Analysis, 33(3):409–422, 1998.

J. Vecer and M. Xu.
Pricing Asian options in a semimartingale model.
Quantitative Finance, 4(2):170–175, 2004.

Outline

Lemma 1

Proof. (sketch)

Considering the logarithm of both sides, we see that we need to establish the a.s. convergence of the infinite series

$$\ln(X(\alpha,\beta)) = \sum_{n\geq 1} \ln\left(B_{(\alpha_n,\beta_n-\alpha_n)}\frac{\beta_n}{\alpha_n}\right).$$

Lemma 1 cont.

Proof. (sketch) cont.

The Mellin transform of a beta random variable is given by

$$\mathbb{E}\left[\left(B_{(a,b)}\right)^{s-1}\right] = \frac{\Gamma(a+b)\Gamma(a+s-1)}{\Gamma(a)\Gamma(a+b+s-1)}, \quad \mathrm{Re}(s) > 1-a.$$

By differentiating the above identity twice and setting $s=1$, we find

$$\mathbb{E}\left[\ln(B_{(a,b)})\right] = \psi(a) - \psi(a+b), \quad \mathrm{Var}\left[\ln(B_{(a,b)})\right] = \psi'(a) - \psi'(a+b),$$

where $\psi(z) := \Gamma'(z)/\Gamma(z)$ is the digamma function.

Lemma 1 cont.

Proof. (sketch) cont.

It is known that $f(z) := \ln(z) - \psi(z)$ is a completely monotone function which decreases to zero. This implies that the function $-f'(z) + 1/z \equiv \psi'(z)$, has the same property. We conclude that both series

$$\sum_{n\geq 1} \mathbb{E}\left[\ln\left(B_{(\alpha_n,\beta_n-\alpha_n)}\frac{\beta_n}{\alpha_n}\right)\right] = \sum_{n\geq 1}(f(\alpha_n) - f(\beta_n)),$$

$$\sum_{n\geq 1} \mathrm{Var}\left[\ln\left(B_{(\alpha_n,\beta_n-\alpha_n)}\frac{\beta_n}{\alpha_n}\right)\right] = \sum_{n\geq 1}(\psi'(\alpha_n) - \psi'(\beta_n))$$

converge, therefore Khintchine-Kolmogorov Convergence Theorem implies a.s. convergence of the infinite series. □

Back to the presentation.

The Mellin transform of I_q

Theorem

The Mellin transform $\mathcal{M}(s) := \mathbb{E}[(X(\alpha,\beta))^{s-1}]$ *is an analytic function for* $\mathrm{Re}(s) > 1-\alpha_1$ *with the form*

$$\mathcal{M}(s) = \prod_{n\geq 1} \frac{\Gamma(\beta_n)\Gamma(\alpha_n+s-1)}{\Gamma(\alpha_n)\Gamma(\beta_n+s-1)} \left(\frac{\beta_n}{\alpha_n}\right)^{s-1}.$$

$\mathcal{M}(s)$ *can be analytically continued to a meromorphic function. For* $n \geq 1$ *define* $\tilde{\alpha}_n = \beta_n$ *and* $\tilde{\beta}_n = \alpha_{n+1}$*. The sequences* $\tilde{\alpha}_n$ *and* $\tilde{\beta}_n$ *also satisfy the interlacing property. Define* $\tilde{\mathcal{M}}(s) = \mathbb{E}\left[X(\tilde{\alpha},\tilde{\beta})^{s-1}\right]$*. We have the following identity*

$$\mathcal{M}(s) \times \tilde{\mathcal{M}}(s) = \frac{\Gamma(\alpha_1+s-1)}{\Gamma(\alpha_1)\alpha_1^{s-1}}, \quad \mathrm{Re}(s) > 1-\alpha_1.$$

The Mellin transform of I_q cont.

Using the previous theorem, we can prove the result.

Proof. (sketch)

- ζ, ρ and $\tilde{\zeta}$, $\tilde{\rho}$ interlace so $X(\tilde{\rho}, \tilde{\zeta})$ and $X(\zeta, \rho)$ are well defined by our lemma
- With $M_1(s) := \mathbb{E}[(X(\tilde{\rho}, \tilde{\zeta}))^{s-1}]$ and $M_2(s) := \mathbb{E}[(X(\zeta, \rho))^{s-1}]$ and our theorem we see that the infinite product is the Mellin transform of $C \times \frac{X(\tilde{\rho}, \tilde{\zeta})}{X(\zeta, \rho)}$ and it is is analytic in the strip $0 < \text{Re}(s) < 1 + \zeta_1$.

The Mellin transform of I_q cont.

Proof. (sketch) cont.

- Verification result: A function $f(s)$ is the Mellin transform of I_q if
  - (i) for some $\theta > 0$, the function $f(s)$ is analytic and zero free in the vertical strip $0 < \mathrm{Re}(s) < 1 + \theta$;
  - (ii) the function $f(s)$ satisfies

$$f(s+1) = \frac{s}{q - \psi(s)} f(s), \quad 0 < s < \theta,$$

    where $\psi(s)$ is the Laplace exponent of the process X;
  - (iii) $|f(s)|^{-1} = o(\exp(2\pi|\mathrm{Im}(s)|))$ as $\mathrm{Im}(s) \to \infty$, uniformly in the strip $0 < \mathrm{Re}(s) < 1 + \theta$.

The Mellin transform of I_q cont.

Proof. (sketch) cont.

- We check the verification result for $\mathcal{M}(s) = f(s)$. Point (i) follows if we let $\theta = \zeta_1$ and look at $\mathcal{M}(s)$.
- Point (ii) follows from the form of $\mathcal{M}(s)$ and $q - \psi(z)$.

The Mellin transform of I_q cont.

Proof. (sketch) cont.

- Point (iii) follows from the following facts:
  - According to our theorem, we can rewrite $\mathcal{M}(s)^{-1}$ as

$$C^{1-s}\frac{\Gamma(\zeta_1)\zeta_1^{1-s}}{\Gamma(s)\Gamma(\zeta_1+1-s)} \times \tilde{M}_1(s) \times \tilde{M}_2(2-s), \quad 0 < \text{Re}(s) < 1+\zeta_1,$$

    where $\tilde{M}_1(s) := \mathbb{E}[(X(\alpha,\beta))^{s-1}]$ and $\tilde{M}_2(s) := \mathbb{E}[(X(\tilde{\alpha},\tilde{\beta}))^{s-1}]$ for shifted sequences α, β, $\tilde{\alpha}$, and $\tilde{\beta}$.
  - $|\tilde{M}_i(s)| < \tilde{M}_i(\text{Re}(s))$ by the properties of Mellin transform and both $\tilde{M}_i$ are analytic on $0 \le \text{Re}(s) \le 1+\zeta_1$ so therefore bounded.

The Mellin transform of I_q cont.

Proof. (sketch) cont.

- Point (iii) follows from the following facts (cont.):
  - Now using the fact that the following limit exists uniformly in x

$$\lim_{y\to\infty} |\Gamma(x+\mathrm{i}y)|e^{\frac{\pi}{2}|y|}|y|^{\frac{1}{2}-x} = \sqrt{2\pi}, \quad x, y \in \mathbb{R},$$

we have the result

$$\frac{1}{|\Gamma(s)\Gamma(\zeta_1+1-s)|} = o(\exp((\pi+\epsilon)|\mathrm{Im}(s)|))$$

as $\mathrm{Im}(s) \to \infty$ for any $\epsilon > 0$.

□

Back to the presentation.

Truncating $\mathcal{M}(s)$

Note that the function $R_N(s) = \mathbb{E}[(\epsilon^{(N)})^{s-1}]$ is analytic in the strip $-\hat{\rho}_N < \mathrm{Re}(s) < 1 + \zeta_{N+1}$, therefore the moments m_k are finite for all $k < 1 + \zeta_{N+1}$. Using the functional equation $\mathcal{M}(s+1) = s\mathcal{M}(s)/(q - \psi(s))$, we find

$$m_k = \mathbb{E}[(\epsilon^{(N)})^k] = R_N(k+1) = \frac{\mathcal{M}(k+1)}{\mathcal{M}_N(k+1)}$$
$$= \frac{k!}{\mathcal{M}_N(k+1)} \prod_{j=1}^{k} \frac{1}{q - \psi(j)}.$$

Back to the presentation.

Numerics: Details

Method 1: We observe that $h(k,q) < +\infty$ for $q > r$, since $\mathbb{E}\left[(I_q - k)^+\right] < \mathbb{E}[I_q] = (q-r)^{-1}$. Further, for $q > r$ we can also establish that $\zeta_1 > 1$. Since $\mathcal{M}(s)$ is finite on $\mathrm{Re}(s) \in (0, 1+\zeta_1)$ we see the following expression is finite on the (non-empty) strip $0 < \mathrm{Re}(s) < \zeta_1 - 1$.

$$
\begin{aligned}
\int_0^\infty h(k,q) k^{s-1} \mathrm{d}k &= \mathbb{E}\left[\int_0^\infty (I_q - k)^+ \, k^{s-1} \mathrm{d}k\right] \\
&= \mathbb{E}\left[\int_0^{I_q} (I_q - k) \, k^{s-1} \mathrm{d}k\right] \\
&= \frac{\mathbb{E}\left[I_q^{s+1}\right]}{s(s+1)} = \frac{\mathcal{M}(s+2,q)}{s(s+1)}
\end{aligned}
$$

Method 1 (cont.): Now, using the approximation of $\mathcal{M}(s)$ with 10, 20, 40, or 80 terms we calculate $h(k,q)$ as the inverse Mellin transform

$$h(k,q) = \frac{k^{-d_1}}{2\pi} \int_{\mathbb{R}} \frac{\mathcal{M}(d_1 + \mathrm{i}v + 2, q)}{(d_1 + \mathrm{i}v)(d_1 + \mathrm{i}v + 1)} e^{-\mathrm{i}v \ln(k)} \mathrm{d}v,$$

where $d_1 \in (0, \zeta_1 - 1)$. From here we calculate $f(k,t)$ via the inverse Laplace transform, which can be written as the cosine transform

$$f(k,t) = \frac{2e^{d_2 t}}{\pi} \int_0^\infty \mathrm{Re}\left[\frac{h(k, d_2 + \mathrm{i}u)}{d_2 + \mathrm{i}u}\right] \cos(ut) \mathrm{d}u,$$

where $d_2 > r$. We evaluate the oscillatory integrals via Filon's method with 400 discretization points using domain of integration $-100 < v < 100$ and $0 < u < 200$ respectively.

Method 2: We approximate our process by a hyper-exponential process. In particular we approximate the Laplace exponent by a function having finite sums instead of infinite series,

$$\tilde{\psi}(z) = \frac{1}{2}\tilde{\sigma}^2 z^2 + \tilde{\mu} z + z^2 \sum_{n=1}^{N} \frac{a_n}{\rho_n(\rho_n - z)} + z^2 \sum_{n=1}^{N} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)}.$$

Method 2 (cont.): The Mellin transform of $\tilde{I}_q$ for this process can be calculated exactly as

$$\tilde{\mathcal{M}}(s,q) = \mathbb{E}\left[\tilde{I}_q^{s-1}\right] = a \times \left(\frac{\tilde{\sigma}^2}{2}\right)^{1-s} \times \Gamma(s) \times \frac{\prod_{j=1}^{N} \Gamma(\hat{\rho}_j + s)}{\prod_{j=1}^{N+1} \Gamma(\hat{\zeta}_j + s)} \times \frac{\prod_{j=1}^{N+1} \Gamma(1 + \zeta_j - s)}{\prod_{j=1}^{N} \Gamma(1 + \rho_j - s)},$$

where $a = a(q)$ is chosen so that $\tilde{\mathcal{M}}(1,q) = 1$. Proceed as in Method 1.

Method 3: Monte-Carlo simulation. We approximate the theta-process $X = \{X_t\}_{0\le t\le T}$ by a random walk $Z = \{Z_n\}_{0\le n\le 400}$ with $Z_0 = 0$ and the increment $Z_{n+1} - Z_n \stackrel{d}{=} X_{T/400}$. The price of the Asian option is approximated then by the following expectation

$$e^{-rT}\mathbb{E}\left[\left(\frac{1}{400}\sum_{n=1}^{400} S_0 e^{Z_n} - K\right)^+\right],$$

which we estimate by sampling 10^6 paths of the random walk.

Method 3 (cont.): In order to sample from the distribution of $Y := Z_{n+1} - Z_n$, we compute its density $p_Y(x)$ via the inverse Fourier transform

$$p_Y(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathbb{E}\left[e^{\mathrm{i}zY}\right] e^{-\mathrm{i}zx} \mathrm{d}z,$$

where $\mathbb{E}\left[e^{\mathrm{i}zY}\right] = \mathbb{E}\left[e^{\mathrm{i}zX_{T/400}}\right] = \exp\left((T/400)\psi(\mathrm{i}z)\right)$. Again, in order to compute the inverse Fourier transform, we use Filon's method with 10^6 discretization points.

Back to the presentation.