

Analytical methods for Lévy processes with applications to finance

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Joint work with Alexey Kuznetsov.

- 1 Introduction
- 2 Transform methods for option pricing
- 3 Some families of Lévy processes
- 4 Asian options and the exponential functional of a meromorphic Lévy process
- 5 Approximating Lévy processes with completely monotone jumps

Overview

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- Show the natural connection to the Laplace exponent, the Wiener-Hopf factors, and the exponential functional
- Introduce (review) two families of “analytically tractable” Lévy processes
- Demonstrate two new results for pricing a variety of options using the existing integral transform methods and these two families

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European Call

Compute

$$E(A_0, K, T) := e^{-rT} \mathbb{E}[(A_T - K)^+],$$

equivalently, compute

$$f(k) := \mathbb{E}[(A_T - e^k)^+],$$

where $A_t = A_0 \exp(X_t)$ is the stock price, T is the expiry, r is the discount rate, K is the strike price, and $k = \log(K)$.

European Call (cont.)

Carr and Madan 1999: Consider the Laplace transform of $f(k)$

$$\phi(z) = \int_{\mathbb{R}} f(k) e^{zk} dk = \frac{\mathbb{E}[A_t^{z+1}]}{z(z+1)} = A_0^{z+1} \frac{e^{t\psi(z+1)}}{z(z+1)},$$

where $\psi(z)$ is the *Laplace exponent* of X , i.e.

$$\psi(z) := \frac{1}{t} \log \mathbb{E}[e^{zX_t}], \quad z \in i\mathbb{R}.$$

We see that if $\psi(z)$ can be identified explicitly, then we have an explicit expression for $\phi(z)$.

Barrier Options: Down-and-out Put

Compute

$$D(A_0, K, B, T) := e^{-rT} \mathbb{E} \left[(K - A_T)^+ \mathbb{I} \left(\inf_{0 \leq t \leq T} A_t > B \right) \right],$$

equivalently, compute

$$f(t) := \mathbb{E}[(k - e^{X_t})^+ \mathbb{I}(\underline{X}_t > b)],$$

where $0 < B < A_0$ is the barrier, $\underline{X}_t := \inf_{0 \leq s \leq t} X_s$ is the running infimum process, $k := K/S_0$, and $b := \log(B/A_0)$.

Barrier Options: Down-and-out Put (cont.)

Jeannin and Pistorius 2010 : Consider the Laplace transform of $f(t)$

$$F(q) := q \int_{\mathbb{R}^+} e^{-qt} f(t) dt = \mathbb{E}[(k - e^{X_{\mathbf{e}(q)}})^+ \mathbb{I}(\underline{X}_{\mathbf{e}(q)} > b)],$$

where $\mathbf{e}(q)$ is an exponential random variable, independent of X , which has mean q^{-1} .

Why is this useful?

Aside: The Wiener-Hopf factorization

- Define $S_q := \overline{X}_{e(q)}$ and $I_q := \underline{X}_{e(q)}$

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-

$$\frac{q}{q - \psi(z)} = \phi_q^+(-z)\phi_q^-(z), \quad z \in i\mathbb{R},$$

since

$$\frac{q}{q - \psi(z)} = \mathbb{E} \left[e^{zX_{e(q)}} \right] = \mathbb{E} \left[e^{z(X_{e(q)} - S_q) + zS_q} \right]$$

Back to the down-and-out put

Using the Wiener-Hopf factorization we may rewrite $F(q)$ as

$$\begin{aligned} F(q) &= \mathbb{E}[(k - e^{X_{\mathbf{e}(q)}})^+ \mathbb{I}(\underline{X}_{\mathbf{e}(q)} > b)] \\ &= \mathbb{E}[(k - e^{S_q + I_q})^+ \mathbb{I}(I_q > -b)]. \end{aligned}$$

Now, if we know the Wiener-Hopf factors explicitly, or even better, if we know the distributions of I_q and S_q we can work out a semi-explicit, or even explicit expression for $F(q)$. Lookback options can be treated similarly.

Perpetual American put

Compute

$$v(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E}[e^{-r\tau}(K - e^{x+X_\tau})^+],$$

where \mathcal{T} is the set of all stopping times for the filtration generated by X . Then for Regular Lévy processes of exponential type (e.g. NIG, and CGMY) we have (Boyarchenko and Levendorskii 2002 and Alili and Kyprianou 2005)

$$\int_{\mathbb{R}} e^{izx} v(x) dx = K \frac{e^{izx^*}}{iz(iz+1)} \phi_q^-(iz), \quad z \in \mathbb{R},$$

where

$$x^* = \log(K) \phi_q^-(1).$$

Asian call

Compute

fwd

$$C(A_0, K, T) := e^{-rT} \mathbb{E} \left[\left(\frac{1}{T} \int_0^T A_0 e^{X_u} du - K \right)^+ \right],$$

equivalently compute

$$f(k, t) := \mathbb{E} \left[\left(\int_0^t e^{X_u} du - k \right)^+ \right].$$

.

Asian call (cont.)

Taking the Laplace transform we get

$$h(k, q) := q \int_{\mathbb{R}^+} e^{-qt} f(k, t) dt = \mathbb{E} \left[\left(\int_0^{e(q)} e^{X_t} dt - k \right)^+ \right].$$

Why is this useful?

The object $I_{e(q)} := \int_0^{e(q)} e^{X_t} dt$ is known as the *exponential functional* of the Lévy process X . It is a well-studied object with applications outside of mathematical finance. Also, there exist several known methods for determining its distribution.

Asian call (cont.)

If the distribution of $I_{\mathbf{e}(q)}$ is known, and tractable enough, then we can determine $h(k, q)$ explicitly (Yor and Geman 1993). If not, we can transform $h(k, q)$ again

$$\begin{aligned}\Phi(z, q) &:= \int_{\mathbb{R}^+} h(k, q) k^{z-1} dk = \mathbb{E} \left[\int_{\mathbb{R}^+} (I_{\mathbf{e}(q)} - k)^+ k^{z-1} dk \right] \\ &= \mathbb{E} \left[\int_0^{I_{\mathbf{e}(q)}} (I_{\mathbf{e}(q)} - k) k^{z-1} dk \right] = \frac{\mathbb{E} [I_{\mathbf{e}(q)}^{z+1}]}{z(z+1)} = \frac{\mathcal{M}(I_{\mathbf{e}(q)}, z+2)}{z(z+1)},\end{aligned}$$

where $\mathcal{M}(I_{\mathbf{e}(q)}, z) := \mathbb{E}[I_{\mathbf{e}(q)}^{z-1}]$ is the Mellin transform of $I_{\mathbf{e}(q)}$. This approach was pioneered by Cai and Kou 2010. If we can find an explicit expression for $\mathcal{M}(I_{\mathbf{e}(q)}, z)$, then we have an explicit expression for $\Phi(z, q)$.

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Analytically tractable families

We have seen, that the transform techniques depend on three key objects: the Laplace exponent, the Wiener-Hopf factors, and the exponential functional. Naturally, we ask: Are there Lévy processes for which we have explicit expressions for all three?

Yes, for example:

- Processes with jumps of rational transform
 - Processes with phase-type jumps
 - Hyper-exponential processes
- Meromorphic processes

Example: Hyper-exponential process

The density of the Lévy measure is

fwd

$$\pi(x) = \mathbb{I}(x > 0) \sum_{n=1}^N a_n \rho_n e^{-\rho_n x} + \mathbb{I}(x < 0) \sum_{i=1}^{\hat{N}} \hat{a}_n \hat{\rho}_n e^{\hat{\rho}_n x},$$

where all the coefficients are positive.

Example: Hyper-exponential process

The Laplace exponent is a rational function

$$\psi(z) = \frac{\sigma^2 z^2}{2} + \mu z + z^2 \sum_{n=1}^{\hat{N}} \frac{\hat{a}_n}{\hat{\rho}_n(\hat{\rho}_n + z)} + z^2 \sum_{n=1}^N \frac{a_n}{\rho_n(\rho_n - z)},$$

and the (real) solutions ζ_n and $-\hat{\zeta}_n$ of $\psi(z) = q$ and the poles of $\psi(z)$ satisfy the important *interlacing property*

$$0 < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 \dots$$

$$0 < \hat{\zeta}_1 < \hat{\rho}_1 < \hat{\zeta}_2 < \hat{\rho}_2 \dots$$

Example: Hyper-exponential process

Assume $\sigma > 0$

fwd

- The Wiener-Hopf factors are given by

$$\phi_q^+(z) = \frac{1}{1 + \frac{z}{\zeta_1}} \prod_{n=1}^N \frac{1 + \frac{z}{\rho_n}}{1 + \frac{z}{\zeta_{n+1}}}, \quad \phi_q^-(z) = \frac{1}{1 + \frac{z}{\zeta_1}} \prod_{n=1}^{\hat{N}} \frac{1 + \frac{z}{\hat{\rho}_n}}{1 + \frac{z}{\zeta_{n+1}}},$$

- The distribution of S_q is a mixture of exponentials

$$\frac{d}{dx} \mathbb{P}(S_q \leq x) = \sum_{n=1}^{N+1} c_n \zeta_n e^{-\zeta_n x},$$

where $c_n > 0$ and $\sum c_n = 1$, and similarly for I_q .

Example: Hyper-exponential process

Define

$$\mathcal{G}(z) := \frac{\prod_{n=1}^{N+1} \Gamma(\zeta_k - z + 1)}{\prod_{n=1}^N \Gamma(\rho_n - z + 1)} \times \frac{\prod_{n=1}^{\hat{N}} \Gamma(\hat{\rho}_n + z)}{\prod_{k=1}^{\hat{N}+1} \Gamma(\hat{\zeta}_k + z)},$$

then

$$\mathcal{M}(I_{\mathbf{e}(q)}, z) = \left(\frac{\sigma^2}{2}\right)^{1-z} \times \Gamma(z) \times \frac{\mathcal{G}(z)}{\mathcal{G}(1)},$$

and

$$I_{\mathbf{e}(q)} \stackrel{d}{=} \frac{2}{\sigma^2} \frac{B_{(1, \hat{\zeta}_1)}}{G_{(\zeta_{N+1}, 1)}} \frac{\prod_{n=1}^{\hat{N}} B_{(\hat{\rho}_n + 1, \hat{\zeta}_{n+1} - \hat{\rho}_n)}}{\prod_{n=1}^N B_{(\zeta_n, \rho_n - \zeta_n)}}.$$

Example: Hyper-exponential process

It seems like Hyper-exponential processes have everything we want, however, they have one serious disadvantage: hyper-exponential processes are necessarily finite activity processes, and for applications in finance we often want infinite activity processes, sometimes maybe even infinite variation.

Meromorphic processes

Meromorphic processes are the generalization of hyper-exponential processes. Essentially, a meromorphic process results from replacing the finite sum in the Lévy density of a hyper-exponential process by an infinite series.

recall

Everything has an “infinite” analogue which has precisely the expected form.

| | Jump activity | Laplace exponent | Interlacing property | Wiener-Hopf factors | Exp. functional |
|--------------|---------------|------------------|----------------------|---------------------|------------------------------|
| Hyper-exp. | Finite | Rational | Yes | Finite product | Finite product of r.v.'s |
| Mero-morphic | Any | Mero-morphic | Yes | Infinite product | Infinite product of r.v.'s * |

Popular stock price models

Popular processes used to model log-stock prices in the literature include the Variance gamma (VG), CGMY/KoBoL, and NIG processes. These have the desired infinite jump activity property, but they are cumbersome for pricing some of the path dependent options: *none have explicit Wiener-Hopf factorizations, and the distribution of the exponential functional is unknown for all.* So what can we do?

- 1 Use meromorphic processes
- 2 Approximate by meromorphic processes
- 3 Approximate by hyper-exponential processes? **

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Other pricing methods

The complication is that we have a path dependent option and $Z_t = A_0 \int_0^t e^{X_u} du$ is not a Markov process.

recall

There are other pricing methods: Monte Carlo, Brownian Motion (Milevsky and Posner 1998 & Shreve 2004), IDE (Semimartingale Vecer and Xu 2004 & Jump diffusion Bayraktar and Xing 2011).

However, I have not seen any papers which price these options in the general setting of processes with two-sided jumps and infinite activity/infinite variation other than with Monte Carlo methods.

The distribution of $I_{e(q)}$

- Kai and Cou 2010 for the hyper-exponential case (finite activity jumps)
- Kuznetsov 2012 for processes of jumps of rational transform (finite activity jumps)
- A. Kuznetsov and J.C. Pardo, 2013 for hyper-geometric processes (infinite activity jumps but distribution is known for only one value of q)

Products of Beta random variables

With any two unbounded sequences $\alpha = \{\alpha_n\}_{n \geq 1}$ and $\beta = \{\beta_n\}_{n \geq 1}$ which satisfy the interlacing property

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \beta_3 \dots$$

we associate an infinite product of independent beta random variables, defined as

$$J(\alpha, \beta) := \prod_{n \geq 1} B_{(\alpha_n, \beta_n - \alpha_n)} \frac{\beta_n}{\alpha_n}.$$

Lemma

$J(\alpha, \beta)$ converges a.s.

Main Result

Theorem

Assume that $q > 0$. Define $\hat{\rho}_0 := 0$ and the four sequences

$$\zeta := \{\zeta_n\}_{n \geq 1}, \rho := \{\rho_n\}_{n \geq 1}, \tilde{\zeta} := \{1 + \hat{\zeta}_n\}_{n \geq 1}, \tilde{\rho} := \{1 + \hat{\rho}_{n-1}\}_{n \geq 1}.$$

Then we have the following identity in distribution

$$I_{\mathbf{e}(q)} \stackrel{d}{=} C(q) \times \frac{J(\tilde{\rho}, \tilde{\zeta})}{J(\zeta, \rho)},$$

where $C(q)$ is a constant and the random variables $J(\tilde{\rho}, \tilde{\zeta})$ and $J(\zeta, \rho)$ are independent. cont. \rightarrow

Main Result cont.

Theorem cont.

The Mellin transform $\mathcal{M}(I_{\mathbf{e}(q)}, z)$ is finite for $0 < \operatorname{Re}(z) < 1 + \zeta_1$ and is given by

$$\mathcal{M}(I_{\mathbf{e}(q)}, z) = C^{s-1} \overbrace{\prod_{n \geq 1} \frac{\Gamma(\hat{\zeta}_n + 1) \Gamma(\hat{\rho}_{n-1} + z)}{\Gamma(\hat{\rho}_{n-1} + 1) \Gamma(\hat{\zeta}_n + z)} \left(\frac{\hat{\zeta}_n + 1}{\hat{\rho}_{n-1} + 1} \right)^{z-1}}^{\mathcal{M}(J(\tilde{\rho}, \tilde{\zeta}), z)} \times \underbrace{\prod_{n \geq 1} \frac{\Gamma(\rho_n) \Gamma(\zeta_n + 1 - z)}{\Gamma(\zeta_n) \Gamma(\rho_n + 1 - z)} \left(\frac{\zeta_n}{\rho_n} \right)^{z-1}}_{\mathcal{M}(J(\zeta, \rho), 2-z)}.$$

A rough idea of the proof

We use the following verification result due to Kuznetsov and Pardo 2013: A function $f(z)$ is the Mellin transform of $I_{\mathbf{e}(q)}$ if

- 1 for some $\theta > 0$, the function $f(z)$ is analytic and zero free in the vertical strip $0 < \operatorname{Re}(z) < 1 + \theta$;
- 2 the function $f(z)$ satisfies

$$f(z+1) = \frac{z}{q - \psi(z)} f(z), \quad 0 < z < \theta,$$

where $\psi(z)$ is the Laplace exponent of the process X ;

- 3 $|f(z)|^{-1} = o(\exp(2\pi|\operatorname{Im}(z)|))$ as $\operatorname{Im}(z) \rightarrow \infty$, uniformly in the strip $0 < \operatorname{Re}(z) < 1 + \theta$.

A rough idea of the proof (cont.)

We need to find a candidate function $f(z)$ and we let point 2 guide us. We are aided by the fact that $q - \psi(z)$ is just a product of linear factors involving the roots and poles recall. Therefore, we are solving many simpler functional equations of the form:

$$f(z+1) = (a \pm z)^k f(z),$$

where a represents a root or a pole, and $k \in \{-1, 1\}$. A solution of such an equation can readily be obtained using the well known formula

$$\Gamma(z+1) = z\Gamma(z),$$

for the gamma function.

Numerics

We will use a theta process for which we have a closed form formula for $\psi(z)$. We can manipulate parameters of the the process to give a process with infinite activity and variation.

Parameter Set I will give a process with a Gaussian component and jumps of infinite activity but finite variation.

Parameter Set II gives a process with zero Gaussian component and jumps of infinite variation.

$A_0 = 100$, $T = 1$, $K = 105$, and $r = 0.03$, with risk neutral condition $\psi(1) = r$ satisfied (this and the assumption $\rho_1 > 1$ ensures key quantities are finite).

Numerics: Pricing an Asian Option Results

| N | Algorithm 1, price | Time (sec.) | Algorithm 2, price | Time (sec.) |
|-----|--------------------|-------------|--------------------|-------------|
| 10 | 4.724627 | 1.6 | 4.720675 | 1.2 |
| 20 | 4.727780 | 2.8 | 4.728032 | 1.8 |
| 40 | 4.728013 | 4.8 | 4.728031 | 3.4 |
| 80 | 4.728029 | 9.2 | 4.728031 | 7.1 |

Table : The price of the Asian option, parameter set I. The Monte-Carlo estimate of the price is 4.7386 with the standard deviation 0.0172. The exact price is $4.72802 \pm 1.0e-5$.

Details

Numerics: Pricing an Asian Option Results

| N | Algorithm 1, price | Time (sec.) | Algorithm 2, price | Time (sec.) |
|-----|--------------------|-------------|--------------------|-------------|
| 10 | 10.620243 | 1.6 | 10.621039 | 1.2 |
| 20 | 10.620049 | 3.0 | 10.620171 | 2.2 |
| 40 | 10.620037 | 4.8 | 10.620054 | 3.6 |
| 80 | 10.620036 | 9.6 | 10.620039 | 7.4 |

Table : The price of the Asian option, parameter set II. The Monte-Carlo estimate of the price is 10.6136 with the standard deviation 0.0251. The exact price is $10.62003 \pm 1.0e-5$.

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Completely monotone jumps

Definition

A function $f(x)$ is called completely monotone if $(-1)^n f^{(n)}(x) > 0$ for all $x > 0$, $n = 0, 1, 2, \dots$

Definition

A Lévy process has completely monotone jumps, if the Lévy measure is absolutely continuous with density $\pi(x)$, and $\pi(x)$ and $\pi(-x)$ are completely monotone for $x \in (0, \infty)$.

Assumption: From now on we assume all processes have completely monotone jumps and $\pi(x)$ decreases exponentially fast as $x \rightarrow \pm\infty$.

Some facts

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- Hyper-exponential processes are dense in the class of completely monotone processes in the sense of weak convergence (Jeannin and Pistorius 2010).

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- All of the processes mentioned satisfy our assumption.
- Hyper-exponential processes are dense in the class of completely monotone processes in the sense of weak convergence (Jeannin and Pistorius 2010).
- The jump density of a process X is completely monotone if, and only if, S_q and I_q are mixtures of exponentials (Rogers 1983).

Main idea

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Main idea

- Approximating a Lévy process is equivalent to approximating its Laplace exponent $\psi(z)$.
- The Laplace exponent of a hyperexponential process is a rational function.
- Thus we have two problems:
 - (1) Approximate $\psi(z)$ by a rational function $\tilde{\psi}(z)$,
 - (2) Show that $\tilde{\psi}(z)$ is itself a Laplace exponent of a Lévy process.

Padé approximation

Definition

Let f be a function with a power series representation $f(z) = \sum_{i=0}^{\infty} c_i z^i$. If there exist polynomials $P_m(z)$ and $Q_n(z)$ satisfying $\deg(P) \leq m$, $\deg(Q) \leq n$, $Q_n(0) = 1$ and

$$\frac{P_m(z)}{Q_n(z)} = c_0 + c_1 z + \cdots + c_{m+n} z^{m+n} + O(z^{m+n+1}), \quad z \rightarrow 0,$$

then we say that $f^{[m/n]}(z) := P_m(z)/Q_n(z)$ is the $[m/n]$ Padé approximant of f .

A simple example of Padé approximations

| $m \backslash n$ | 0 | 1 | 2 | 3 |
|------------------|---|--|---|---|
| 0 | $\frac{1}{1}$ | $\frac{1}{1-z}$ | $\frac{1}{1-z+\frac{1}{2}z^2}$ | $\frac{1}{1-z+\frac{1}{2}z^2-\frac{1}{6}z^3}$ |
| 1 | $\frac{1+z}{1}$ | $\frac{1+\frac{1}{2}z}{1-\frac{1}{2}z}$ | $\frac{1+\frac{1}{3}z}{1-\frac{2}{3}z+\frac{1}{6}z^2}$ | $\frac{1+\frac{1}{4}z}{1-\frac{3}{4}z+\frac{1}{4}z^2-\frac{1}{24}z^3}$ |
| 2 | $\frac{1+z+\frac{1}{2}z^2}{1}$ | $\frac{1+\frac{2}{3}z+\frac{1}{6}z^2}{1-\frac{1}{3}z}$ | $\frac{1+\frac{1}{2}z+\frac{1}{12}z^2}{1-\frac{1}{2}z+\frac{1}{12}z^2}$ | $\frac{1+\frac{2}{5}z+\frac{1}{20}z^2}{1-\frac{3}{5}z+\frac{3}{20}z^2-\frac{1}{60}z^3}$ |
| 3 | $\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3}{1}$ | $\frac{1+\frac{3}{4}z+\frac{1}{4}z^2+\frac{1}{24}z^3}{1-\frac{1}{4}z}$ | $\frac{1+\frac{3}{5}z+\frac{3}{20}z^2+\frac{1}{60}z^3}{1-\frac{2}{5}z+\frac{1}{20}z^2}$ | $\frac{1+\frac{1}{2}z+\frac{1}{10}z^2+\frac{1}{120}z^3}{1-\frac{1}{2}z+\frac{1}{10}z^2-\frac{1}{120}z^3}$ |
| 4 | $\frac{1+z+\frac{1}{2}z^2+\frac{1}{6}z^3+\frac{1}{24}z^4}{1}$ | $\frac{1+\frac{4}{5}z+\frac{3}{10}z^2+\frac{1}{15}z^3+\frac{1}{120}z^4}{1-\frac{1}{5}z}$ | $\frac{1+\frac{2}{3}z+\frac{1}{5}z^2+\frac{1}{30}z^3+\frac{1}{360}z^4}{1-\frac{1}{3}z+\frac{1}{30}z^2}$ | $\frac{1+\frac{4}{7}z+\frac{1}{7}z^2+\frac{2}{105}z^3+\frac{1}{840}z^4}{1-\frac{3}{7}z+\frac{1}{14}z^2-\frac{1}{210}z^3}$ |

Figure : The initial part of the Padé table for e^z

Gaussian quadrature

- ν is a finite positive measure on a closed bounded interval $[a, b]$

Gaussian quadrature

- ν is a finite positive measure on a closed bounded interval $[a, b]$
- For each n we want to find a measure $\tilde{\nu}_n$ on a finite number of points in $[a, b]$ such that we match the first $2n - 1$ moments of ν , i.e.

$$\int_{[a,b]} x^j \nu(dx) = \sum_i^n x_i^j w_i, \quad \text{for } j = 1, \dots, 2n - 1.$$

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- The points x_i and w_i are the nodes and weights of the Gaussian quadrature.

Gaussian quadrature and orthogonal polynomials

- $\{p_n(x)\}_{n \geq 0}$ be the sequence of orthogonal polynomials with respect to the measure $\nu(dx)$: $\deg(p_n) = n$ and

$$(p_n, p_m)_\nu := \int_{[a,b]} p_n(x) p_m(x) \nu(dx) = d_n \delta_{n,m}$$

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$$(p_n, p_m)_\nu := \int_{[a,b]} p_n(x) p_m(x) \nu(dx) = d_n \delta_{n,m}$$

- The nodes of the Gaussian quadrature $\tilde{\nu}_n$ are the zeros of p_n and the weights may be calculated from p_{n-1}, p_n (Szegő 1975).

Lévy-Khintchine representation

The Lévy-Khintchine representation for $\psi(z)$ is

$$\psi(z) = \frac{\sigma^2 z^2}{2} + az + \int_{\mathbb{R}} (e^{zx} - 1 - zx\mathbb{I}(|x| < 1)) \Pi(dx), \quad z \in i\mathbb{R}.$$

Bernstein's Theorem

We can develop a very useful description of the processes which satisfy our assumption using Bernstein's theorem. A process satisfies our assumption if, and only if, there exists a positive measure $\mu(\mathrm{d}u)$, with support in $\mathbb{R} \setminus \{0\}$, such that for all $x \in \mathbb{R}$

$$\pi(x) = \mathbb{I}(x > 0) \int_{(0,\infty)} e^{-ux} \mu(\mathrm{d}u) + \mathbb{I}(x < 0) \int_{(-\infty,0)} e^{-ux} \mu(\mathrm{d}u), \quad (1)$$

and $\mu(\mathrm{d}u)$ assigns no mass to a non-empty interval $(-\hat{\rho}, \rho)$ containing the origin + integrability condition on $\mu(\mathrm{d}u)$.

A change of variables

We define

$$\mu^*(A) := \mu(\{v \in \mathbb{R} : v^{-1} \in A\}).$$

Then, the Lévy-Khintchine formula + Fubini's theorem + change of variables give us

$$\psi(z) = \frac{\sigma^2}{2} z^2 + az + z^2 \int_{[-\hat{\rho}^{-1}, \rho^{-1}]} \frac{|v|^3 \mu^*(dv)}{1 - vz}.$$

Key Observation: $|v|^3 \mu^*(dv)$ is a finite measure, with bounded support.

Main theorem (two-sided case)

Assume that $\sigma = 0$. Let $\{x_i\}_{1 \leq i \leq n}$ and $\{w_i\}_{1 \leq i \leq n}$ be the nodes and the weights of the Gaussian quadrature of order n with respect to the measure $|v|^3 \mu^*(dv)$. We define

$$\psi_n(z) := az + z^2 \sum_{i=1}^n \frac{w_i}{1 - zx_i}.$$

Theorem

- (i) *The function $\psi_n(z)$ is the $[n + 1/n]$ Padé approximant of $\psi(z)$.*
- (ii) *The function $\psi_n(z)$ is the Laplace exponent of a hyperexponential process $X^{(n)}$ having the characteristic triple $(a, \sigma_n^2, \pi_n)_{h \equiv x}$, where*

Main theorem (two-sided case)

Theorem

(ii)

$$\pi_n(x) := \begin{cases} \sum_{1 \leq i \leq n : x_i < 0} w_i |x_i|^{-3} e^{-\frac{x}{x_i}}, & \text{if } x < 0, \\ \sum_{1 \leq i \leq n : x_i > 0} w_i x_i^{-3} e^{-\frac{x}{x_i}}, & \text{if } x > 0. \end{cases}$$

(iii) The random variables $X_1^{(n)}$ and X_1 satisfy $\mathbb{E}[(X_1^{(n)})^j] = \mathbb{E}[(X_1)^j]$ for $1 \leq j \leq 2n + 1$.

Convergence

Theorem

For any compact set $A \subset \mathbb{C} \setminus \{(-\infty, -\hat{\rho}] \cup [\rho, \infty)\}$ there exist $c_1 = c_1(A) > 0$ and $c_2 = c_2(A) > 0$ such that for all $z \in A$ and all $n \geq 1$

$$|\psi_n(z) - \psi(z)| < c_1 e^{-c_2 n}.$$

One-sided processes

- For CM subordinators, all three functions $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.

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One-sided processes

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- For CM spectrally-positive processes of infinite variation, only two functions $\psi^{[n+1/n]}(z)$, $\psi^{[n+2/n]}(z)$ are Laplace exponents of hyperexponential processes.
- There exist explicit formulas for a number of important examples: In the VG case we have $\psi^{[n/n]}(z) = P_n(z)/Q_n(z)$, where

$$P_n(z) = 2 \sum_{j=0}^n \binom{n}{j}^2 [H_{n-j} - H_j] (1-z)^j, Q_n(z) = z^n P_n\left(\frac{2}{z} - 1\right).$$

and $H_j := 1 + 1/2 + \cdots + 1/j$.

How do we prove all these results?

- One can show that only $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$ and $\psi^{[n+2/n]}(z)$ can possibly be Laplace exponents of a Lévy process

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- One can show that only $\psi^{[n/n]}(z)$, $\psi^{[n+1/n]}(z)$ and $\psi^{[n+2/n]}(z)$ can possibly be Laplace exponents of a Lévy process
- The function

$$g(z) = \int_{[-\hat{\rho}^{-1}, \rho^{-1}]} \frac{|v|^3 \mu^*(dv)}{1 - vz}.$$

is closely related to a *Stieltjes function*:

$$f(z) := \int_{[0, R^{-1}]} \frac{\nu(du)}{1 + zu}$$

Some more theory on Stieltjes functions.

- $f^{[m/n]}(z)$ always exists provided $m \geq n - 1$.

(Baker and Graves-Morris 1996 & Allen et. al 1975)

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$$f^{[n-1/n]}(z) = \frac{(-z)^{n-1} q_{n-1}(-1/z)}{(-z)^n p_n(-1/z)} = \sum_{i=1}^n \frac{w_i}{1 + x_i z}.$$

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- Plus convergence results

(Baker and Graves-Morris 1996 & Allen et. al 1975)

Math Finance applications

We will work with the following two processes: the VG process V defined by the Laplace exponent

$$\psi(z) = \mu z - c \log \left(1 - \frac{z}{\rho} \right) - c \log \left(1 + \frac{z}{\hat{\rho}} \right),$$

and parameters

$$(\rho, \hat{\rho}, c) = (21.8735, 56.4414, 5.0),$$

and the CGMY process Z defined by the Laplace exponent

$$\psi(z) = \mu z + C\Gamma(-Y) [(M - z)^Y - M^Y + (G + z)^Y - G^Y],$$

and parameters

$$(C, G, M, Y) = (1, 8.8, 14.5, 1.2).$$

European call

| | two-sided $[2N + 1/2N]$ | one-sided $[N + 1/N]$ | one-sided $[N + 2/N]$ |
|---------|----------------------------|--------------------------|--------------------------|
| $N = 1$ | -2.75e-2 | 1.93e-2 | -3.72e-3 |
| $N = 2$ | -4.86e-6 | -4.19e-6 | 9.5e-5 |
| $N = 3$ | 4.80e-7 | -1.48e-5 | -2.54e-7 |
| $N = 4$ | 2.9e-8 | 6.41e-7 | -1.55e-7 |
| $N = 5$ | 1.14e-9 | 5.58e-9 | 6.95e-9 |

Table : The error in computing the price of the European call option for the CGMY Z-model. Initial stock price is $A_0 = 100$, strike price $K = 100$, maturity $T = 0.25$ and interest rate $r = 0.04$. The benchmark price is 11.9207826467.

Down-and-out put

We calculate barrier option prices for the process V , for four values $A_0 \in \{81, 91, 101, 111\}$ and with other parameters given by $K = 100$, $B = 80$, $r = 0.04879$ and $T = 0.5$

| | $A_0 = 81$ | $A_0 = 91$ | $A_0 = 101$ | $A_0 = 111$ |
|-----------|------------|------------|-------------|-------------|
| Benchmark | 3.39880 | 7.38668 | 1.40351 | 0.04280 |
| $N = 2$ | 3.44551 | 7.39225 | 1.40527 | 0.04233 |
| $N = 4$ | 3.40209 | 7.38957 | 1.40329 | 0.04258 |
| $N = 6$ | 3.39910 | 7.38939 | 1.40332 | 0.04258 |
| $N = 8$ | 3.39856 | 7.38936 | 1.40332 | 0.04258 |
| $N = 10$ | 3.39853 | 7.38936 | 1.40332 | 0.04258 |

Table : Barrier option prices calculated for the VG process V -model. Benchmark prices obtained from Kudryavtsev and Levendorskiĭ 2009



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To approximate $\mathcal{M}(I_{\mathbf{e}(q)}, z)$ we can simply truncate our infinite product, but convergence may be slow. The more terms we need, the more roots $-\tilde{\zeta}_n$ and ζ_n we need to calculate which is computationally expensive. Note if we truncate the transform we get:

$$\mathcal{M}_N(z) := a_N \times b_N^{z-1} \times \prod_{n=1}^N \frac{\Gamma(\hat{\rho}_{n-1} + z)}{\Gamma(\hat{\zeta}_n + z)} \frac{\Gamma(\zeta_n + 1 - z)}{\Gamma(\rho_n + 1 - z)}$$

where a_N and b_N are normalizing constants.

Now we note that

$$\mathcal{M}(I_{\mathbf{e}(q)}, z) = \mathcal{M}_N(z) R_N(z)$$

where $R_N(z) = \mathcal{M}(I_{\mathbf{e}(q)}, z) / \mathcal{M}_N(z)$ is the Mellin transform of the tail of our product of beta random variables which we denote $\epsilon^{(N)}$.

Instead of simply letting $R_N(z) = 1$ we try to find a random variable ξ matching the first two moments m_1 and m_2 of $\epsilon^{(N)}$.

We let ξ be a beta random variable of the second kind which has density:

$$\mathbb{P}(\xi \in dx) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} y^{a-1} (1+y)^{-a-b} dy, \quad y > 0.$$

We choose $a, b > 0$ such $\mathbb{E}[\xi] = m_1$ and $\mathbb{E}[\xi^2] = m_2$.

Method 1: Using the approximation of $\mathcal{M}(I_{\mathbf{e}(q)}, z)$ with 10, 20, 40, or 80 terms we calculate $h(k, q)$ as the inverse Mellin transform

$$h(k, q) = \frac{k^{-d_1}}{2\pi} \int_{\mathbb{R}} \frac{\mathcal{M}(d_1 + iv + 2, q)}{(d_1 + iv)(d_1 + iv + 1)} e^{-v \ln(k)} dv,$$

where $d_1 \in (0, \zeta_1 - 1)$. From here we calculate $f(k, t)$ via the inverse Laplace transform, which can be written as the cosine transform

$$f(k, t) = \frac{2e^{d_2 t}}{\pi} \int_{\mathbb{R}_+} \operatorname{Re} \left[\frac{h(k, d_2 + iu)}{d_2 + iu} \right] \cos(ut) du,$$

where $d_2 > r$. We evaluate the oscillatory integrals via Filon's method with 400 discretization points using domain of integration $-100 < v < 100$ and $0 < u < 200$ respectively.

Method 2: We approximate our process by a hyper-exponential process. In particular we approximate the Laplace exponent by a function having finite sums instead of infinite series.

Method 3: Monte-Carlo simulation. We approximate the theta-process $X = \{X_t\}_{0 \leq t \leq T}$ by a random walk $Z = \{Z_n\}_{0 \leq n \leq 400}$ with $Z_0 = 0$ and the increment $Z_{n+1} - Z_n \stackrel{d}{=} X_{T/400}$. The price of the Asian option is approximated then by the following expectation

$$e^{-rT} \mathbb{E} \left[\left(\frac{1}{400} \sum_{n=1}^{400} A_0 e^{Z_n} - K \right)^+ \right],$$

which we estimate by sampling 10^6 paths of the random walk.

Method 3 (cont.): In order to sample from the distribution of $Y := Z_{n+1} - Z_n$, we compute its density $p_Y(x)$ via the inverse Fourier transform

$$p_Y(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathbb{E} [e^{izY}] e^{-izx} dz,$$

where $\mathbb{E} [e^{izY}] = \mathbb{E} [e^{izX_{T/400}}] = \exp((T/400)\psi(iz))$. In order to compute the inverse Fourier transform, we use Filon's method with 10^6 discretization points.

Back to the [presentation](#).